1 Generating functions, continued

1.1 Generating functions and partitions

We can make use of generating functions to answer some questions a bit more restrictive than we've done so far:

Question 1: *Find a generating function for the number of ways to distribute n balls among 3 boxes if the first box can contain any number of balls, the second box contains an odd number of balls, and the third contains a multiple of 5 less than 15.*

Answer 1: *The procedure for filling the first box is known to be represented by the g.f.* $1+x+x^2+x^3+\cdots = \frac{1}{1-x}$*. The second, however, would be* $x+x^3+x^5+x^7+\cdots = x(1+x^2+x^4+x^6+\cdots) = \frac{x}{1-x^2}$*, and the third would be* $1+x^5+x^{10} = \frac{1-x^{15}}{1-x^5}$*. In total, we would have* $\frac{x-x^{15}}{(1-x)(1-x^2)(1-x^5)}$*, which is not tremendously amenable to simplification, but we could actually do it with partial fractions! (this one was done with computer assistance)*

$$\frac{3}{2(1-x)^2} - \frac{33}{4(1-x)} - \frac{1}{4(1+x)} + 7 + 6x + 5x^2 + 4x^3 + 3x^4 + 2x^5 + 2x^6 + x^7 + x^8$$

The three terms at the beginning of this expansion expand to $\sum_{n=0}^{\infty} \frac{3}{2}\binom{n+2-1}{2-1}x^n - \frac{33}{4}x^n - \frac{1}{4}(-x)^n$

So for $x > 8$*, the coefficient of* x^n *is* $\frac{3}{2}(n+1) - \frac{33}{4} - \frac{(-1)^n}{4}$*.*

Note that we could've phrased this another way:

Question 2: *Find a generating function for the number of nonnegative integer solutions to* $x_1 + (2x_2+1) + 5x_3 = n$*, with* $5x_3 < 30$*.*

which means we can now produce generating functions — if not simple answers — to the number of ways to divide any n into linear combinations.

Here's one such question which has a certain amount of everyday relevance:

$$x_1 + 5x_2 + 10x_3 + 25x_4 + 50x_5 + 100x_6 = n$$

A nonnegative integer solution to this is equivalent to a way of making change for n cents in standard American change (including the half-dollar and dollar coin). The generating function for this can be worked out to be

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})(1-x^{100})}$$

Trying to actually calculate this, even with partial fractions, is not recommended. However, using partial sums and computer algebra systems, we can find values for individual entries:

Question 3: *How many ways are there to make change for a dollar?*

Answer 3: *From the above, we can get an approximation sufficient for workng out the first hundred terms of the sequence:*

$$(1+x+x^2+\cdots+x^{100})(1+x^5+x^{10}+\cdots+x^{100})(1+x^{10}+x^{20}+\cdots+x^{100})(1+x^{25}+x^{50}+x^{75}+x^{100})(1+x^{50}+x^{100})(1+x^{100})$$

which gives us a 600th-degree polynomial: $1+x+x^2+x^3+x^4+2x^5+\cdots+293x^{100}+\cdots+x^{600}$*. We can thus see that there are 293 ways to make change for a dollar.*

This all brings us towards integer partitions. Above the question was really how to divide n into 1s, 5s, 10s, 25s, 50s, and 100s. If we look at integer partitions, we can see that we're really dividing a number up into several "value classes", e.g. we recall that $p(5) = 7$ with the partitions 5, $4+1$, $3+2$, $3+1+1$, $2+1+1+1+1$, $2+2+1$, and $1+1+1+1+1$. If we allow x_i to represent the number of is used in a partition, we can see that these partitions are in a one-to-one correspondence with solutions to $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 5$, whcih would be the x^5 coefficient of the generating function $\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$.

This gets more troublesome if we wanted a generating function for *arbitrary* partition numbers, but we can at least present a symbolic form, if not an actual formula:

$$\sum_{n=0} \infty p_n = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots} = \frac{1}{\prod_{i=1}^{\infty}(1-x^i)}$$

And specific formulas good enough for finding specific coefficients can be produced by using partial products and partial sums.

Some other neat partition problems:

Question 4: *How many ways are there to express n as a sum of* distinct *integers?*

Answer 4: *This is, pleasantly enough, an easier question than the unrestricted partition! Instead of counting the number of each value i that appears with the expression $1+x^i+x^{2i}+x^{3i}+\cdots = \frac{1}{1-x^i}$, we instead have each number either appearing once or not at all, so the expression to represent this selection is $1+x^i$. Thus our total generating function is*

$$(1+x)(1+x^2)(1+x^3)(1+x^4)+\cdots = \prod_{i=1}^{\infty}(1+x^i)$$

and the answer to the question above is the coefficient of x^n (which does not have a closed form, but is amenable to fairly easy symbolic calculation).

Question 5: *How many ways are there to express n as a sum of integers no greater than k?*

Answer 5: *Here, instead of an infinite number of sub-processes, we only need k of them, so we have for once a generating function which is a finite product:*

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots(1-x^k)}$$

Now we modify this slightly in a way that will be useful for us in actually returning to our permutation statistics.

Question 6: *How many ways are there to express n as sum of integers no greater than k, one of which is k itself?*

Answer 6: *This is as above, except instead of the kth process being $1+x^k+x^{2k}+\cdots = \frac{1}{1-x^k}$, we have $x^k+x^{2k}+\cdots = \frac{x^k}{1-x^k}$, so the total generating function is*

$$\frac{x^k}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots(1-x^k)}$$

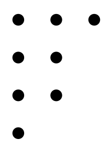
This will actually bring us to the purpose of creating a generating function for the one statistic which we handwaved when building the twelve-fold way. We could note by observation that the number of ways to partition n into nonzero parts of which the largest is k is equal to the number of ways to partition n into k nonzero parts, as in this example for $n = 8$ and $k = 3$:

$$\begin{aligned} 3+1+1+1+1+1 &\leftrightarrow 6+1+1 \\ 3+2+1+1+1 &\leftrightarrow 5+2+1 \\ 3+2+2+1 &\leftrightarrow 4+3+1 \\ 3+3+1+1 &\leftrightarrow 4+2+2 \\ 3+3+2 &\leftrightarrow 3+3+2 \end{aligned}$$

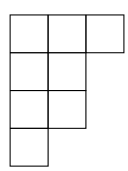
To exhibit this relationship, we have recourse to a visual technique for presenting partitions:

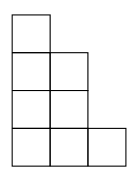
Definition 1. The *Ferrers diagram* for the partition $a_1 + a_2 + a_3 + \cdots + a_k$ for $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k > 0$ consists of k left-justified rows of equally-spaced dots with a_i dots in the ith row, for each i.

For instance, here we have a Ferrers diagram for $3 + 2 + 2 + 1$:



An alternative to the Ferrers diagram is a figure drawn with boxes instead of dots. Such a figure is called a *Young diagram*, and can be conventionally drawn in one of two styles: with the largest row on top (in what is known as the *English-style*, shown below on the left) or on bottom (which is *French-style*, shown below on the right).





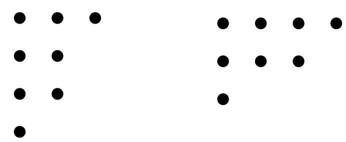
Any of these visualizations will work equally well. There are several useful geometrically-determined properties of Ferrers diagrams, listed without proof below (the proofs are largely straightforward results of the definition of the Ferrers diagram). Assuming a Ferrers diagram is associated with the partition $a_1 + a_2 + \cdots + a_k$ for $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k > 0$, we have two particularly useful geometric properties.

- The Ferrers diagram has a height of k (i.e. a total of k rows).
- The diagram has a width of a_1 (i.e. a total of a_1 columns).
- A Ferrers diagram uniquely determines, and is uniquely determined by, a partition.

Thus, we may conclude that $p_k(n)$ is the number of Ferrers diagrams with n dots of height k; in Question 5 we determined the number of Ferrers diagrams of width k, and above we hinted that these might be the same quantity. In fact we can find an explicit bijection:

Definition 2. The *transpose* or *conjugate* of a Ferrers diagram is a diagram produced by exchanging the rows and columns of the diagram. Alternatively, the *conjugate* of a partition $a_1+a_2+a_3+\cdots+a_k$ with $a_1 \geq a_2 \geq \cdots \geq a_k > 0$ is a partition $b_1+b_2+b_3+\cdots+b_{a_1}$, where $b_i = \|\{j : a_j \geq i\}|$.

The second definition above is a direct definition of a partition conjugate without appeal to geometric intuition. Below is an exhibit of conjugation applied to a Ferrers diagram, or alternatively to the associated partition:



$3+2+2+1$ $4+3+1$

Note that two successive conjugations return the original Ferrers diagram; note also that distinct diagrams have distinct conjugates. Since conjugation swaps height and width, conjugation illustrates that partitions into k parts, and partitions with one part of size k and other parts of size $\leq k$ are in fact equinumerous, and since we had a generating function for the latter, we can use it for the former, giving us a generating function for $p_k(n)$:

$$\sum_{n=0}^{\infty} p_k(n)x^n = \frac{x^k}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots(1-x^k)}$$

In addition, if we were to conjugate a partition with a maximum entry of k, we would get a partition with a maximum length of k; this would correspond to a twelvefold way statistic without surjection, which we know to be $p_0(n)+p_1(n)+p_2(n)+\cdots+p_k(n)$, which gives us the not entirely obvious algebraic identity:

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots(1-x^k)} = 1 + \sum_{i=1}^{k} \frac{x^i}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots(1-x^i)}$$

1.2 Exponential generating functions

So far, we've looked at algebraic representations for distribution of indistinct objects, and we've used that to shed a great deal of light on two rows of our twelvefold way. Can we do the same for distinct objects?

Suppose we have two processes for distributing n distinct objects; we can think of them as sequences $(a_0, a_1, a_2, a_3, \ldots)$ and $(b_0, b_1, b_2, b_3, \ldots)$.

We might ask what the sequence is for distributing n objects using one or the other (but not both) of these processes: this would be $(a_0+b_0, a_1+b_1, a_2+b_2, \ldots)$. There are plenty of algebraic structures where this would be simple addition.

How about distribution using these processes alternatingly? When the objects were indistinguishable, recall that this was $(a_0b_0, a_0b_1+a_1b_0, a_0b_2+a_1b_1+a_2b_0, \cdots)$. However, here we must not only decide how many objects to deal with using process A and how many using process B, but in addition *which* objects we deal with using each process.

If we incorporate this information, then we have $(1, a_1+b_1, a_2+2a_1b_1+b_2, a_3+3a_1b_2+3a_2b_1+b_3)$. The way to dispose of n objects will end up being $a_0b_n + \binom{n}{1}a_1b_{n-1} + \cdots + \binom{n}{i}a_ib_{n-i} + \cdots a_nb_0$. This is actually fairly sensible. We can split our n objects up as i dealt with using process A and $n-i$ using process B in $\binom{n}{i}$ ways; then there are a_i ways to place the i selected objects with process A and b_{n-i} ways to place the selected objects with process B.

But now we ask: what useful algebraic object has this property, that multiplying two associated with the sequences $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$ gives a product associated with the sequence $(a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + 2a_1b_1 + a_2b_0, \ldots)$?

The answer is a surprising tweak on the generating functions we have seen to date:

Definition 3. If a_n enumerates a number of ways to place, select, or otherwise arrange n distinct objects, then $f(x) = \sum_{n=0}^{\infty} \frac{a_nx^n}{n!}$ is called the *exponential generating function* for a_n.

Now, note that multiplication of two generating functions in fact produces the desired result:

$$\begin{aligned}\left(\sum_{n=0}^{\infty}\frac{a_n}{n!}x^n\right)\left(\sum_{n=0}^{\infty}\frac{b_n}{n!}x^n\right) &= \sum_{n=0}^{\infty}\left(\frac{a_0}{0!}\frac{b_n}{n!} + \frac{a_1}{1!}\frac{b_{n-1}}{(n-1)!} + \cdots + \frac{a_n}{n!}\frac{b_0}{0!}\right)x^n \\ &= \sum_{n=0}^{\infty}\left(\frac{n!}{0!n!}a_0b_n + \frac{n!}{1!(n-1)!}a_1b_{n-1} + \cdots + \frac{n!}{n!0!}a_nb_0\right)\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\binom{n}{0}a_0b_n + \binom{n}{1}a_1b_{n-1} + \cdots + \binom{n}{n}a_nb_0\right)\frac{x^n}{n!}\end{aligned}$$

So this allows us to construct generating functions from several independent distribution procedures for distinct items. As in the case of the ordinary generating function, let's start by trying this out with a statistic we already know:

Question 7: *Find an exponential generating function to enumerate the ways to place n distinct balls in r boxes, so that no box contains more than one ball.*

Answer 7: *Let us start by noting that the enumeration statistic for such a placement is known to be $n!\binom{r}{n}$, and we will judge our procedure a success if it yields tihs expression as coefficients.*

The generating funciton for putting balls in a single box will be $1\left(\frac{x^0}{0!}\right) + 1\left(\frac{x^1}{1!}\right)$. We may either place no balls in it (in only one way), or place the only labeled ball we have on hand in it (in only one way).

The concatenation of r such procedures can be performed by multiplying this quantity by itself r times, so our exponential generating function here would be $(1+x)^r$. If we use the binomial theorem on this, we would get that the generating function is $\sum_{n=0}^{r}\binom{r}{n}x^n$; however, in an exponential generating function we look at coefficients not of x^n but of $\frac{x^n}{n!}$; it is thus expedient to rewrite this form as $\sum_{n=0}^{r} n!\binom{r}{n}\left(\frac{x^n}{n!}\right)$.

We can also find ways to symbolically express and answer questions which would have been insanely difficult before:

Question 8: *If we have 3 boxes, each of which can contain no more than 3 balls, how many ways are there to distribute n labeled balls among them?*

Answer 8: *Each box can be filled with either zero balls in one way, 1 ball in one way, 2 balls in one way, or 3 balls in one way: thus this procedure has generating function*

$$1\left(\frac{x^0}{0!}\right)+1\left(\frac{x^1}{1!}\right)+1\left(\frac{x^2}{2!}\right)+1\left(\frac{x^3}{3!}\right)=1+x+\frac{x^2}{2}+\frac{x^3}{6}$$

And concatenation of the three procedures gives

$$\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)^3=1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3+\frac{13}{4}x^4+\frac{7}{4}x^5+\frac{17}{24}x^6+\frac{5}{24}x^7+\frac{1}{24}x^8+\frac{1}{216}x^9$$
$$=1+3x+9\frac{x^2}{2}+27\frac{x^3}{6}+78\frac{x^4}{4!}+210\frac{x^5}{5!}+510\frac{x^6}{6!}+1050\frac{x^7}{7!}+1680\frac{x^8}{8!}+1680\frac{x^9}{9!}$$

The coefficients on $\frac{x^n}{n!}$ for n from 0 to 9 gives the number of ways to distribute n distinct balls (for $n>9$, of course, such a distribution is impossible, so there are zero ways to do so).

Of course, the real fun comes when our functions are series instead of polynomials! As is often the case, we'll start with a question to which we already know the answer.

Question 9: *Find an exponential generating function to enumerate the ways to place n distinct balls in r boxes, if each box can contain any number of balls.*

Answer 9: *As before, we know this enumeration statistic, and it is r^n, so we have a reasonable expectation of getting that expression appearing in the coefficients of the generating function.*

The procedure for filling one box is, as we saw in the previous examples, the sum of monomials indicating that we can put i labeled balls in the box in exactly one way, that is, $1\frac{x^i}{i!}$. So, adding up all such monomials, we get that the exponential generating function associated with putting balls into a single box is:

$$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots$$

which is a series we can describe succinctly with a function – this is the well-known Taylor series for e^x, so the exponential generating function for the procedure of putting n distinct balls in a single box freely is e^x! Since we have r boxes to fill, the generating function for filling all x of them is the product of r copies of this generating function, that is, $(e^x)^r=e^{rx}$. Re-expressing this as a series, we see that the generating function for this entire procedure is

$$e^{rx}=\sum_{n=0}^{\infty}\frac{(rx)^n}{n!}=\sum_{n=0}^{\infty}r^n\frac{x^n}{n!}$$

which meets our expectations.

We've filled in two cells of row 1 of the twelvefold way. Let's try to do the third — which we originally needed inclusion-exclusion to handle!

Question 10: *Find an exponential generating function to enumerate the ways to place n distinct balls in r boxes, if each box must contain at least one ball.*

Answer 10: *The procedure for filling one box is the sum of the same terms as were used in the previous example, but we now forbid the possibility of leaving the box empty, so our exponential generating function associated with putting balls into a single box will be*

$$x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots=(e^x-1)$$

so the exponential generating function for the process of processing r boxes in this way is $(e^x - 1)^r$. However, if we want to get this exponential generating function into a state where individual coefficients can be calculated, we'll need to expand this a bit further.

$$\begin{aligned}
(e^x - 1)^r &= \sum_{i=0}^{r} \binom{r}{i} (e^x)^i (-1)^{r-i} \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} e^{ix} \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \sum_{n=0}^{\infty} i^n \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} i^n \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} i^n \right) \frac{x^n}{n!}
\end{aligned}$$