1 Generating functions, continued

1.1 Exponential generating functions and set-partitions

At this point, we've come up with good generating-function discussions based on 3 of the 4 rows of our twelvefold way. Will our integer-partition tricks work with exponential generating functions to give us rules for putting distinct balls into indistinct boxes (a.k.a. set-partitions)?

Recall that for the integer-partition problem our series of sequential procedures was to first select the number of boxes with one ball, then the number of boxes with 2 balls, then the number of boxes with 3 balls, etc. Would that work in this case? Let's try it:

Question 1: *Find an exponential generating function for the number of ways to distribute n balls among unlabeled boxes.*

Answer 1: *An exponential generating function for the number of ways to select boxes with 1 ball in is easy: there is only one way to fill zero boxes, one way to fill one box, etc., so the generating function would be:*

$$1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Filling boxes with two balls gets harder, however: there is one way to set up zero boxes with two balls each, one way to set up one box, three ways to set up two boxes, and fifteen ways to set up six, etc. This ends up being ugly pretty fast, and doesn't lend itself to computation.

However, since we already know this statistic (these are the Bell numbers), we shouldn't have to go home empty-handed: we can actually work out what the generating function is! Recall that

$$B_n = \sum_{k=1}^{n} S(n,k) = \sum_{k=1}^{n} \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n$$

So the generating function we seek is the heinous expression:

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=1}^{n} \sum_{i=0}^{k} \frac{(-1)^{k-i}\binom{k}{i} i^n}{n!k!} x^n$$

This is actually pretty easy! We can find the generating function for $S(n,k)$ from the known generating function for $k!S(n,k)$:

$$\sum_{n=0}^{\infty} k!S(n,k)\frac{x^n}{n!} = (e^x - 1)^k$$

so

$$\sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}$$

and since $B_n = \sum_{k=0}^{n} S(n,k)$, and we can include the zero terms when $k > n$ to get $B_n =$

$\sum_{k=0}^{\infty} S(n,k)$, *then:*

$$\begin{aligned}\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} S(n,k)\frac{x^n}{n!}\\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!}\\ &= \sum_{k=0}^{\infty} \frac{(e^x-1)^k}{k!} = e^{(e^x-1)}\end{aligned}$$

As far as I know, there is no "from first principles" argument to derive this same elegant and simple form.

1.2 Proving binomial identities with generating functions

We can prove binomial identities with generating functions, if we know the GFs of the various binomial expressions. For instance, here's one from the last problem set: **Question 2:** *Prove that* $\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$.

Answer 2: *If we can show that the generating functions* $\sum_{n=0}^{\infty}\sum_{i=0}^{n} i\binom{n}{i}x^n$ *and* $\sum_{n=0}^{\infty} n2^{n-1}x^n$ *are equal, then we're done, since two GFs are equivalent if and only if they're equal in all coefficients.*

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{i=0}^{n} i\binom{n}{i}x^n &= \sum_{i=0}^{\infty} i \sum_{n=0}^{\infty}\binom{n}{i}x^n\\ &= \sum_{i=0}^{\infty} i \sum_{n=-i}^{\infty}\binom{n+i}{i}x^{n+i}\\ &= \sum_{i=0}^{\infty} ix^i \sum_{n=-i}^{\infty}\binom{n+i}{i}x^{n}\\ &= x^{-1}\sum_{i=0}^{\infty} i\left(\frac{x}{1-x}\right)^{i+1}\end{aligned}$$

There is a fairly straightforward simplification of $\sum_{i=0}^{\infty} iy^{i+1}$*! Note that since* $\sum_{i=0}^{\infty} y^i = \frac{1}{1-y}$, *differeintiating both sides with respect to* y *gives* $\sum_{i=0}^{\infty} iy^{i-1} = \frac{1}{(1-y)^2}$, *so* $\sum_{i=0}^{\infty} iy^{i+1} = \frac{y^2}{(1-y)^2}$. *Thus:*

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n} i\binom{n}{i}x^n = x^{-1}\left(\frac{\frac{x}{1-x}}{1-\frac{x}{1-x}}\right)^2 = \frac{x}{(1-2x)^2}$$

Now let's look at $\sum_{n=0}^{\infty} n2^{n-1}x^n$:

$$\begin{aligned}\sum_{n=0}^{\infty} n2^{n-1}x^n &= x\sum_{n=0}^{\infty} 2^{n-1}nx^{n-1} \\ &= x\sum_{n=0}^{\infty} 2^{n-1}\frac{d}{dx}x^n \\ &= x\frac{d}{dx}\sum_{n=0}^{\infty} 2^{n-1}x^n \\ &= x\frac{1}{2}\frac{d}{dx}\sum_{n=0}^{\infty}(2x)^n \\ &= x\frac{1}{2}\frac{d}{dx}\frac{1}{1-2x} = \frac{x}{(1-2x)^2}\end{aligned}$$

1.3 Other strange and wonderful generating function variants

We've been producing generating functions for individual values of k as n varies. In ways this seems a bit artificial: surely a two-parameter enumeration statistic should have a single generating function which differentiates along both parameters? We can do that by using another variable along with x, e.g. a variable y whose exponent indicates the number of boxes used.

Question 3: *How could we produce a function $f(x, y)$ such that the coefficient of x^ny^r represents the number of ways to distribute n identical balls among r distinct boxes with no more than one ball per box?*

Answer 3: *We of course know pre-emptively that the answer is $\sum_{r\geq 0}\sum_{n\geq 0}\binom{r}{n}x^ny^r = \sum_{r\geq 0}(x+1)^ry^r = \frac{1}{1-(x+1)y}$. But how could we derive this from some sort of assembly procedure? Consider a procedure for filling a single box: this procedure uses one box and one ball in one way, and one box and no balls in one way, so its generating function is $1x^0y^1+1x^1y^1 = y+xy$. We may use anywhere from zero to an arbitrarily high number of boxes, so we add together the results of performing this rpocedure any number of times, i.e.:*

$$(y+xy)^0 + (y+xy)^1 + (y+xy)^2 + \cdots = \frac{1}{1-(y+xy)}$$

.

By and large we do not actually use multivariable generating functions, although in algebraic combinatorics they are used to record the number of permutations or partitions with very specific characteristics.

2 Recurrence relations

We might ask a number of not obviously related questions:

- How many ways are there to cover a $1 \times n$ checkerboard with dominoes and with checkers?

- How many bitstrings of length $n-1$ do not have "00" as a substring?
- How many permutations π of the numbers $\{1, 2, \ldots, n\}$ have all $|\pi(i) - i| = 1$?

We might notice, idly, that these three situations are equinumerous. If we experiment a bit more, we find that for successive values of n we get a familiar-looking sequence: 1,1,2,3,5,8,13.... This is the Fibonacci sequence, which we've surely encountered before.

The Fibonacci sequence is the simplest interesting example we have of a *recurrence relation*: it is given by the recursive formula $F_{n+1} = F_n + F_{n-1}$. We can exhibit why this formula models the counts above.

Now, as to solving recurrence relations: the Fibonacci sequence is of an unusual type, where F_n is a linear combination of prior F_i.

Definition 1. A sequence $\{a_n\}$ is given by a *linear homogeneous recurrence relation of order k* if $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}$ for all $n \geq k$.

2.1 Linear homogeneous recurrences solved with linear algebra

The solution method for linear homogeneous equations falls out of three useful propositions, two of which we present immediately and solve easily:

Proposition 1. *If $\{a_n\}$ and $\{b_n\}$ are given by the same LHRR, then so is the linear combination $\{qa_n + rb_n\}$. In other words, linear combinations of an LHRR's solutions are themselves solutions.*

Proof. Given the LHRR $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}$ satisfied by the sequences $\{a_n\}$ and $\{b_n\}$, it follows that

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k} \text{ for } n \geq k$$

and

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + c_3 b_{n-3} + \cdots + c_k b_{n-k} \text{ for } n \geq k$$

Multiplying the first equation by q on both sides and the second by r, and adding the results:

$$\begin{aligned} qa_n + rb_n &= (qc_1 a_{n-1} + qc_2 a_{n-2} + \cdots + qc_k a_{n-k}) + (rc_1 b_{n-1} + rc_2 b_{n-2} + \cdots + rc_k b_{n-k}) \text{ for } n \geq k \\ &= c_1(qa_{n-1} + rb_{n-1}) + c_2(qa_{n-2} + rb_{n-2}) + \cdots + c_k(qa_{n-k} + rb_{n-k}) \text{ for } n \geq k \end{aligned}$$

so that $\{qa_n + rb_n\}$ satisfies the aforementioned LHRR. □

Proposition 2. *A sequence is uniquely determined by an LHRR of order k and the initial values $a_0, a_1, a_2, \ldots, a_{k-1}$.*

Proof. Consider two sequences $\{a_n\}$ and $\{b_n\}$ which both satisfy the LHRR $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}$. Clearly, if $a_0 \neq b_0$, or $a_1 \neq b_1$, or so forth up to $a_{k-1} \neq b_{k-1}$, then the sequences, differing in at least one element, are definitionally distinct.

In contrast, if $a_0 = b_0$, $a_1 = b_1$, and so forth up to $a_{k-1} = b_{k-1}$, then there is a simpler inductive argument that the two sequences are equivalent. We shall inductively show that $a_n = b_n$ for all n. The base cases $n = 0, \ldots, k-1$ are all part of our original presumed equalities. Then, for the unductive step, in proving that $a_n = b_n$, we may assume that $a_i = b_i$ for $i < n$, and in particular we can assert equality between the linear combinations as such:

$$c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k} = c_1 b_{n-1} + c_2 b_{n-2} + c_3 b_{n-3} + \cdots + c_k b_{n-k}$$

and thus, since the LHRR gives the left and right sides of this equation to be a_n and b_n respectively, it follows that $a_n = b_n$. □

These two propositions shed a great deal of light on the nature of a LHRR's solution set: the first proposition tells us that it is a vector space, since it is closed under lienar combinations, and the second proposition, although it is not immediately obvious at present, actually tells us the dimensions of the vector space.

Theorem 1. *For a given LHRR of order k, there are sequences $\{e_n^0\}, \{e_n^1\}, \ldots, \{e_n^{k-1}\}$ such that any sequence $\{a_n\}$ satisfying the LHRR may be expressed as a linear combination of the sequences $\{e_n^i\}$.*

Proof. Let $\{e_n^0\}$ be the sequence (known to be unique from the proposition above) satisfying the LHRR with $e_0^0 = 1$ and $e_1^0 = e_2^0 = \cdots = e_{k-1}^0 = 0$. Likewise, let $\{e_n^1\}$ be given by the initial conditions $e_1^1 = 1$ and $e_0^1 = e_2^1 = \cdots = e_{k-1}^1 = 0$, and so forth such that each $\{e_k^i\}$ is given by $e_i^i = 1$ and $e_j^i = 0$ for $0 \leq j \leq k-1$ and $j \neq i$.

Now, given a sequence $\{a_n\}$ satisfying the LHRR, let us construct a sequence $\{b_n\}$ according to the rule $b_n = e_n^0 a_0 + e_n^1 a_1 + \cdots + e_n^{k-1} a_{k-1}$. Since b_n is a linear combination of sequences satisfying the LHRR, it satisfies the LHRR. And furthermore, we can note:

$$\begin{aligned}
b_0 &= e_0^0 a_0 + e_0^1 a_1 + \cdots + e_0^{k-1} a_{k-1} = 1a_0 + 0a_1 + \cdots + 0a_{k=1} = a_0 \\
b_1 &= e_1^0 a_0 + e_1^1 a_1 + \cdots + e_1^{k-1} a_{k-1} = 0a_0 + 1a_1 + \cdots + 0a_{k=1} = a_1 \\
&\vdots \\
b_{k-1} &= e_{k-1}^0 a_0 + e_{k-1}^1 a_1 + \cdots + e_{k-1}^{k-1} a_{k-1} = 0a_0 + 0a_1 + \cdots + 1a_{k=1} = b_{k-1}
\end{aligned}$$

So, since $\{a_n\}$ and $\{b_n\}$ both satisfy the same k-order LHRR and have equal terms for the first k elements, they are equivalent. Thus $\{a_n\}$ is expressible as a linear combination of the $\{e_n^i\}$ sequences. □

The above argument further shows that any nontrivial lienar combination of the $\{e_n^i\}$ sequences is nonzero, which in linear-algebraic terms means that the sequences $\{e_n^i\}$ are linearly independent. Since they form both an independent set and a spanning set for the set of solutions to the LHRR, we can assert that the above-mentioned set of sequences is not only a spanning set for the space of LHRR-satisfying sequences, but is in fact a *basis* for this space (so the space has dimension k). Furthermore, linear-algebraic knowledge about vector spaces tells us that, once we know the dimension k of a space, *any* set of k linearly independent vectors will be a basis of the space. Thus, we have a new goal: *to find closed-form expressions for k linearly independent sequences satisfying a LHRR of order k.*

We have a proposition which will aid us immeasurably in this task:

Proposition 3. *For a nonzero number r, $a_n = r^n$ is a solution to the LHRR $a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + \cdots + c_ka_{n-k}$ iff r is a root of the polynomial (called the* characteristic polynomial*) $x^k - c_1x^{k-1} - c_2x^{k-2} - c_3x^{k-3} - \cdots - c_kx^0$.*

Proof. Suppose $a_n = r^n$. We shall show equivalency between $\{a_n\}$ satisfying the LHRR $a_n = c_1a_{n-1}+c_2a_{n-2}+c_3a_{n-3}+\cdots+c_ka_{n-k}$ and r satisfying $r^k-c_1r^{k-1}-c_2r^{k-2}-c_3r^{k-3}-\cdots-c_kr^0 = 0$. This is actually extremely simple algebraic manipulation. Supposing $\{a_n\}$ satisfies the LHRR:

$$\begin{aligned}
a_n &= c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + \cdots + c_ka_{n-k} \\
r^n &= c_1r^{n-1} + c_2r^{n-2} + c_3r^{n-3} + \cdots + c_kr^{n-k} \\
r^{n-k}r^k &= c_1r^{n-k}r^{k-1} + c_2r^{n-k}r^{k-2} + c_3r^{n-k}r^{k-3} + \cdots + c_kr^{n-k}r^0 \\
r^k &= c_1r^{k-1} + c_2r^{k-2} + c_3r^{k-3} + \cdots + c_kr^0 \\
r^k - c_1r^{k-1} - c_2r^{k-2} - c_3r^{k-3} - \cdots - c_kr^0 &= 0
\end{aligned}$$

The converse follows by performing the same process in reverse. □

In the case where the characteristic polynomial has k distinct roots, this is sufficient to completely determine the space of sequences satisfying the recurrence relation, and then, via algebra on the initial condition, find a closed form for a particular sequence. We can see how this procedure would work, using the Fibonacci nubmers as an example:

Question 4: *What is a closed form for the sequence given by $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$?*

Answer 4: *We shall start by finding two linearly independent solutions to the second-order LHRR $a_n = a_{n-1} + a_{n-2}$. It has solutions of the form $a_n = r^n$ when r is a root of the characteristic polynomial $x^2 - x - 1$. We know this polynomial has roots $\frac{1\pm\sqrt{5}}{2}$, so that the above LHRR has 2 easy-to-find solutions: $a_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$ and $a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$. These are linearly independent, so this set of 2 linearly independent solutions to the linear homogeneous recurrence relation is a basis to the solution space, so that F_n is a linear combination of them. Thus, there are some k and ℓ such that:*

$$F_n = k\left(\frac{1-\sqrt{5}}{2}\right)^n + \ell\left(\frac{1+\sqrt{5}}{2}\right)^n$$

We may determine k and ℓ by ensuring the linear combination yields the correct initial conditions:

$$\begin{aligned}
1 = F_0 &= k\left(\frac{1-\sqrt{5}}{2}\right)^0 + \ell\left(\frac{1+\sqrt{5}}{2}\right)^0 = k + \ell \\
1 = F_1 &= k\left(\frac{1-\sqrt{5}}{2}\right)^1 + \ell\left(\frac{1+\sqrt{5}}{2}\right)^1 = \left(\frac{1-\sqrt{5}}{2}\right)k + \left(\frac{1+\sqrt{5}}{2}\right)\ell
\end{aligned}$$

This system of equations can be solved comparatively easily (if arithmetically messily) to give $k = \frac{5-\sqrt{5}}{10}$ and $\ell = \frac{5+\sqrt{5}}{10}$, so the closed form for Fibonacci numbers is:

$$F_n = \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

The same line of arguments will work on any linear homogeneous recurrence relation, as long as its characteristic polynomial has distinct roots.

2.2 Linear homogeneous recurrences solved with ordinary generating functions

We can justify the same result, if we like, by building an ordinary generating function for the Fibonacci numbers. Suppose $f(x) = \sum_{n=0}^{\infty} F_n x^n$. Then, what can we say about $f(x)$, and can it give us a closed form for coefficients in its power-series representation?

We may begin by noting that $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Since this equality holds for all $n \geq 0$, it can be phrased as an equality of power series:

$$\sum_{n=0}^{\infty} F_{n+2} x^n = \sum_{n=0}^{\infty} (F_{n+1} + F_n) x^n$$

And we can now manipulate these algebraically until we get several copies of $f(x)$:

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{n+2} x^n &= \sum_{n=0}^{\infty} (F_{n+1} + F_n) x^n \\
x^2 \sum_{n=0}^{\infty} F_{n+2} x^n &= x^2 \sum_{n=0}^{\infty} F_{n+1} x^n + x^2 \sum_{n=0}^{\infty} F_n x^n \\
\sum_{n=0}^{\infty} F_{n+2} x^{n+2} &= x \sum_{n=0}^{\infty} F_{n+1} x^{n+1} + x^2 \sum_{n=0}^{\infty} F_n x^n \\
\sum_{n=2}^{\infty} F_n x^n &= x \sum_{n=1}^{\infty} F_n x^n + x^2 \sum_{n=0}^{\infty} F_n x^n \\
\sum_{n=0}^{\infty} F_n x^n - F_1 x - F_0 &= x \sum_{n=0}^{\infty} F_n x^n - x F_0 + x^2 \sum_{n=0}^{\infty} F_n x^n \\
f(x) - F_1 x - F_0 &= x f(x) - x F_0 + x^2 f(x) \\
(1 - x - x^2) f(x) &= F_1 x + F_0 - x F_0 \\
f(x) &= \frac{F_1 x + F_0 - x F_0}{1 - x - x^2} = \frac{1}{1 - x - x^2}
\end{aligned}$$

A digression: we could have gotten this same generating function using traditional generating-function-production methods on one of the combinatorial objects enumerated by the Fibonacci numbers:

Question 5: *Determine a generating function for the number of ways to cover a $1 \times n$ checkerboard with dominoes and checkers.*

Answer 5: *We could lay down zero objects, which would suffice to cover zero squares in one way; this possibility will contribute $1x^0 = 1$ towards the generating function. If we lay down a single object, that object could be a checker, covering a single square in one way, or a domino, covering two squares in one way. Thus, the prospect of laying down a single object can be represented as $(1x^1 + 1x^2) = x + x^2$. If we lay down two objects, we can enumerate the possible results of doing so by multiplying the generating function for a single placement by itself; that is, $(x + x^2)^2$. Likewise,* *if we place i objects, we can represent the generating function representing the number of ways to cover squares with this as $(x + x^2)^i$. Since we could plausible place any nubmer of objects, our generating function results from adding together all these cases:*

$$1 + (x + x^2) + (x + x^2)^2 + (x + x^2)^3 + \cdots = \frac{1}{1 - (x + x^2)}$$

which is, notably, equivalent to our above generating function for the Fibonacci numbers.

If we were to apply our ordinary-generating-function-producing methodology as above to an arbitrary LHRR, the same procedure would work, albeit with some messy generalities. So if we let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with a_n satisfying an LHRR of order k, then:

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+k}x^n &= \sum_{n=0}^{\infty}(c_1 a_{n+k-1} + c_2 a_{n+k-2} + \cdots + c_k a_n)x^n \\
x^k \sum_{n=0}^{\infty} a_{n+k}x^n &= c_1 x^k \sum_{n=0}^{\infty} a_{n+k-1}x^n + c_2 x^k \sum_{n=0}^{\infty} a_{n+k-2}x^n + \cdots + c_k x^k \sum_{n=0}^{\infty} a_n x^n \\
\sum_{n=0}^{\infty} a_{n+k}x^{n+k} &= c_1 x \sum_{n=0}^{\infty} a_{n+k-1}x^{n+k-1} + c_2 x^2 \sum_{n=0}^{\infty} a_{n+k-2}x^{n+k-2} + \cdots + c_k x^k \sum_{n=0}^{\infty} a_n x^n \\
\sum_{n=k}^{\infty} a_n x^n &= c_1 x \sum_{n=k-1}^{\infty} a_n x^n + c_2 x^2 \sum_{n=k-2}^{\infty} a_n x^n + \cdots + c_k x^k \sum_{n=0}^{\infty} a_n x^n \\
\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{k-1} a_n x^n &= c_1 x \sum_{n=0}^{\infty} a_n x^n - c_1 x \sum_{n=0}^{k-2} a_n x^n + c_2 x^2 \sum_{n=0}^{\infty} a_n x^n - c_2 x^2 \sum_{n=0}^{k-3} a_n x^n + \cdots + c_k x^k \sum_{n=0}^{\infty} a_n x^n \\
f(x) - \sum_{n=0}^{k-1} a_n x^n &= c_1 x f(x) - c_1 x \sum_{n=0}^{k-2} a_n x^n + c_2 x^2 f(x) - c_2 x^2 \sum_{n=0}^{k-3} a_n x^n + \cdots + c_k x^k f(x) \\
(1 - c_1 x - \cdots - c_k x^k) f(x) &= \sum_{n=0}^{k-1} a_n x^n - c_1 x \sum_{n=0}^{k-2} a_n - c_2 x^2 \sum_{n=0}^{k-3} a_n x^n - \cdots - c_{k-1} x^{k-1} a_0 \\
f(x) &= \frac{\sum_{n=0}^{k-1} a_n x^n - c_1 x \sum_{n=0}^{k-2} a_n - c_2 x^2 \sum_{n=0}^{k-3} a_n x^n - \cdots - c_{k-1} x^{k-1} a_0}{1 - c_1 x - c_2 x^2 - \cdots - c_k x^k}
\end{aligned}$$

Note that the numerator of the above expression consists of several finite sums, built from the values $a_0, \ldots, a_{k-1}$, which is to say the initial conditions. The denominator of the function is not quite the characteristic polynomial, but it is in fact a polynomial equivalent to the generating polynomial with coefficients in opposite order; the roots of this polynomial are the reciprocals of the roots of the characteristic polynomial.

The take-home lesson here is that the generating function of a linear homogeneous recurrence relation of order k will always be a rational function, whose denominator has degree k.

And, of course, we have a way to evaluate individual coefficients in a rational generating function, by making use of partial fractions, and we can do so to rediscover the closed form for the Fibonacci numbers:

$$
\begin{aligned}
\frac{1}{1-x-x^2} &= \frac{\frac{\sqrt{5}}{5}}{x+\frac{1+\sqrt{5}}{2}} - \frac{\frac{\sqrt{5}}{5}}{x+\frac{1-\sqrt{5}}{2}} \\
&= \frac{\frac{5-\sqrt{5}}{10}}{\frac{-1+\sqrt{5}}{2}x+1} - \frac{\frac{-5-\sqrt{5}}{10}}{\frac{-1-\sqrt{5}}{2}x+1} \\
&= \frac{\frac{5-\sqrt{5}}{10}}{1-\frac{1-\sqrt{5}}{2}x} + \frac{\frac{5+\sqrt{5}}{10}}{1-\frac{1+\sqrt{5}}{2}x} \\
&= \frac{5-\sqrt{5}}{10}\sum_{n=0}^{\infty}\left(\frac{1-\sqrt{5}}{2}x\right)^n + \frac{5+\sqrt{5}}{10}\sum_{n=0}^{\infty}\left(\frac{1+\sqrt{5}}{2}x\right)^n \\
&= \sum_{n=0}^{\infty}\left[\frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n\right]x^n
\end{aligned}
$$

from which the coefficients can be grabbed — and they agree with our result from the previous section.