1 Recurrence relations, continued

1.1 Linear homogeneous recurrences solved with exponential generating functions

We saw that we could apporach a recurrence with ordinary generating functions. What if, instead, we used exponential generating functions, with $f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$? Let's try it with the Fibonacci nubmers in particular, for discovery purposes:

$$\sum_{n=0}^{\infty} F_{n+2} \frac{x^n}{n!} = \sum_{n=0}^{\infty} (F_{n+1} + F_n) \frac{x^n}{n!}$$

As before, we'll manipulate these until we get several copies of $f(x)$, but not all of our manipulations will be algebraic:

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{n+2} \frac{x^n}{n!} &= \sum_{n=0}^{\infty} (F_{n+1} + F_n) \frac{x^n}{n!} \\
\sum_{n=0}^{\infty} F_{n+2} \frac{d^2}{dx^2} \frac{x^{n+2}}{(n+2)!} &= \sum_{n=0}^{\infty} F_{n+1} \frac{d}{dx} \frac{x^{n+1}}{(n+1)!} + \sum_{n=0}^{\infty} F_n \frac{x^n}{n!} \\
\frac{d^2}{dx^2} \sum_{n=2}^{\infty} F_n \frac{x^n}{n!} &= \frac{d}{dx} \sum_{n=1}^{\infty} F_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} F_n \frac{x^n}{n!} \\
\frac{d^2}{dx^2} \sum_{n=0}^{\infty} F_n \frac{x^n}{n!} &= \frac{d}{dx} \sum_{n=0}^{\infty} F_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} F_n \frac{x^n}{n!} \\
f''(x) &= f'(x) + f(x)
\end{aligned}$$

So our exponential generating function is, surprisingly, the solution to a *differential equation*! We have been skirting differential equation all along with recurrence relations: such phrases as "linear homogeneous", "initial conditions", "characteristic polynomial", and "order k" are more than a little evocative of DE terminology, and here we have a direct relationship between a recurrence and a differential equation. In fact, note that this linear homogeneous differential equation has the same order and characteristic polynomial as the associated LHRR, and even, to a certain point, the same initial conditions: $f(0) = F_0 = 1$, and $f'(0) = F_1 = 1$.

Solving the differential equation $f''(x) = f'(x) + f(x)$ is a matter of identifying its characterstic polynomial $r^2 - r - 1$, and then its roots $r = \frac{1 \pm \sqrt{5}}{2}$.

Thus, we know the exponential generating function $f(x) = ke^{\frac{1-\sqrt{5}}{2}x} + \ell e^{\frac{1+\sqrt{5}}{2}x}$. Since $f(0) = 1$, it follows that $k + \ell = 1$, and since $f'(0) = 1$, it follows that $\frac{1-\sqrt{5}}{2}k + \frac{1+\sqrt{5}}{2}\ell = 1$. This is a system we've solved before to give $k = \frac{5-\sqrt{5}}{10}$ and $\ell = \frac{5+\sqrt{5}}{10}$, so

$$f(x) = \frac{5 - \sqrt{5}}{10} e^{\frac{1-\sqrt{5}}{2}x} + \frac{5 + \sqrt{5}}{10} e^{\frac{1+\sqrt{5}}{2}x}$$

and note that we can get individual coefficients using power-series expansions:

$$\begin{aligned} f(x) &= \frac{5-\sqrt{5}}{10} e^{\frac{1-\sqrt{5}}{2}x} + \frac{5+\sqrt{5}}{10} e^{\frac{1+\sqrt{5}}{2}x} \\ &= \frac{5-\sqrt{5}}{10} \sum_{n=0}^{\infty} \frac{\left(\frac{1-\sqrt{5}}{2}x\right)^n}{n!} + \frac{5+\sqrt{5}}{10} \sum_{n=0}^{\infty} \frac{\left(\frac{1+\sqrt{5}}{2}x\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[\frac{5-\sqrt{5}}{10} \left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{10} \left(\frac{1+\sqrt{5}}{2}\right)^n \right] \frac{x^n}{n!} \end{aligned}$$

We could do the same thing with an arbitrary recurrence relation. If we let $f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$, where $a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \cdots + c_k a_n$, then:

$$\begin{aligned} \sum_{n=0}^{\infty} a_{n+k} \frac{x^n}{n!} &= \sum_{n=0}^{\infty} (c_1 a_{n+k-1} + c_2 a_{n+k-1} + \cdots + c_k a_n) \frac{x^n}{n!} \\ \sum_{n=0}^{\infty} a_{n+k} \frac{d^k}{dx^k} \frac{x^{n+k}}{(n+k)!} &= c_1 \sum_{n=0}^{\infty} a_{n+k-1} \frac{d^{k-1}}{dx^{k-1}} \frac{x^{n+k-1}}{(n+k-1)!} + \cdots + c_k \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \\ \frac{d^k}{dx^k} \sum_{n=k}^{\infty} a_n \frac{x^n}{n!} &= c_1 \frac{d^{k-1}}{dx^{k-1}} \sum_{n=k-1}^{\infty} a_n \frac{x^n}{n!} + \cdots + c_k \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \\ \frac{d^k}{dx^k} \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} &= c_1 \frac{d^{k-1}}{dx^{k-1}} \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + \cdots + c_k \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \\ f^{(k)}(x) &= c_1 f^{(k-1)}(x) + c_2 f^{(k-2)}(x) + \cdots + c_{k-1} f'(x) + c_k f(x) \end{aligned}$$

so the exponential generating function for *any* linear homogeneous recurrence relation is the associated linear homogeneous differential equation. Note also that the initial conditions from the LHRR map to initial conditions for the LHDE straightforwardly: $f(0) = a_0$, $f'(0) = a_1$, $f''(0) = a_2$, and so forth.

1.2 Linear homogeneous recurrences with repeated roots

Recall our original LHRR-solution methodology: find the distinct roots $r_1, r_2, \ldots, r_k$ of the characteristic polynomial, identify every solution as a linear combination of the sequences $\{r_i^n\}$, determine necessary coefficients for a particular solution. However, this method relies on the presumption that the characteristic polynomial has k distinct nonzero roots. This is frequently the case (and even though the roots may be irrational or even complex, this is not intrinsically problematic, as irrational and complex values are subject to the same arithmetic manipulations as any other type of number). The only problem we can have is if there are not k distinct roots, or if roots are zero. The Fundamental Theorem of Algebra guarantees us k roots, but some could be identical or zero.

Zero is not a problem. In order for a LHRR to have order k, the coefficient c_k must be nonzero (otherwise, a_n would not depend on a_{n-k}, so it could be considered as a lower-order LHRR). Thus the constant coefficient in the characteristic polynomial is nonzero, so zero is not a root of the polynomial.

The repeated-root case is subtler, though. It's not much of a problem in the ordinary-generating-function or exponential-generating-function regimes, though, since it simply requires that we know how to deal with repeated roots in the denominator of a partial fraction decomposition or the characteristic polynomial of a differential equation. In fact, we could use either or both of these methods to answer this question preperatory to finding an answer which does not use generating functions.

Question 1: *Find the general form of a sequence satisfying the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$.*

Answer 1: *We might approach this using an ordinary generating function: the methods we determined last week indicate that this would have generating function $f(x) = \frac{P(x)}{1-4x+4x^2} = \frac{P(x)}{(1-2x)^2}$, where $P(x)$ is a linear function determined by the initial conditions. Using known partial-fraction rules, we would decompose this as $f(x) = \frac{k}{1-2x} + \frac{\ell}{(1-2x)^2}$ for k and ℓ determined by $P(x)$, which we could then convert back to*

$$f(x) = k\sum_{n=0}^{\infty}(2x)^n + \ell\sum_{n=0}^{\infty}\binom{n+2-1}{2-1}(2x)^n = \sum_{n=0}^{\infty}(k2^n + \ell n2^n)x^n$$

so the general form of a sequence satisfying the aforementioned recurrence relation is a linear combination of the completely expected $\{2^n\}$ and the somewhat more surprising $\{n2^n\}$.

We might do the same thing with an exponential generating function. If $g(x)$ were an exponential generating function for a sequence satisfying $a_n = 4a_{n-1} - 4a_{n-2}$, then we would know (based on the derivation above) that $g''(x) = 4g'(x) - g(x)$, which would have characteristic polynomial $x^2 - 4x + 4$, which has 2 as a root with multiplicity 2, leading to differential equation solutions e^{2x} and xe^{2x}. As before, we can pull individual coefficients out of this with ease:

$$\begin{aligned}
g(x) &= ke^{2x} + \ell xe^{2x} \\
&= k\sum_{n=0}^{\infty}2^n\frac{x}{n!} + \ell\sum_{n=0}^{\infty}2^n\frac{x^{n+1}}{n!} \\
&= k\sum_{n=0}^{\infty}2^n\frac{x}{n!} + \frac{\ell}{2}\sum_{n=0}^{\infty}2^{n+1}\frac{(n+1)x^{n+1}}{(n+1)!} \\
&= k\sum_{n=0}^{\infty}2^n\frac{x}{n!} + \frac{\ell}{2}\sum_{n=1}^{\infty}2^n\frac{nx^n}{n!} \\
&= k\sum_{n=0}^{\infty}2^n\frac{x}{n!} + \frac{\ell}{2}\sum_{n=0}^{\infty}2^n\frac{nx^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(k2^n + \frac{\ell}{2}n2^n\right)\frac{x^n}{n!}
\end{aligned}$$

We can justify this in our original, blissfully generating-function-free regime. We start with a useful lemma:

Lemma 1. *If a polynomial $P(x)$ has a root r of multiplicity q, then $P'(x)$ has r as a root of multiplicity at least $q-1$. Inductively, it follows that $P^{(i)}(x)$ has r as a root for all $0 \le i \le q-1$.*

Proof. This is a relatively inductive straightforward application of the product rule: $P(x) = (x - r)^q p(x)$ for some polynomial $p(x)$. Then

$$P'(x) = q(x-r)^{q-1}p(x) + (x-r)^q p'(x) = (x-r)^{q-1}[qp(x) + p'(x)]$$

Since $P'(x)$ can be factored into a product of $(x-r)^{q-1}$ and a polynomial, $P'(x)$ has r as a root of multiplicity at least q. □

Theorem 1. *If the characteristic polynomial of the linear homogeneous recurrence relation* $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$ *has root* r *with multiplicity* q*, then* $\{r^n\}, \{nr^n\}, \{n(n-1)r^n\}, \ldots, \{n(n-1)(n-2)\cdots(n-(q-1))r^n\}$ *all satisfy the recurrence relation.*

Proof. Since the characteristic polynomial has root r with multiplicity q, we know that $P(x) = x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k$ has a zero of multiplicity k at r. Likewise, for any n, $P_n(x) = x^{n-k}P(x) = x^n - c_1 x^{n-1} - \cdots - c_k x^{n-k}$ has r as a zero of multiplicity k. By the lemma above, it follows that $P_n^{(i)}(x)$ has r as a zero for $0 \leq i \leq n$. Note that this derivative can be shown to be:

$$P_n^{(i)}(x) = n(n-1)\cdots(n-i+1)x^n - c_1(n-1)(n-2)\cdots(n-i)x^n - \cdots c_k(n-k)(n-k-1)\cdots(n-k-i+1)x^n$$

so if we let $a_n = n(n-1)(n-2)\cdots(n-i+1)r^n$, then we know that

$$0 = P_n^{(i)}(r) = a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k}$$

which is equivalent to the satisfaction criteria for the LHRR, so these expressions do in fact satisfy the LHRR. □

Note that r^n and nr^n satisfying this LHRR guarantee that *any* linear combination $(a + bn)r^n$ will satisfy it; including $n(n-1)r^n$ allows us to isolate a quadratic component, $n(n-1)(n-2)r^n$ allows isolation of a cubic component, and so forth. In fact, $r^n, nr^n, n(n-1)r^n, \ldots, n(n-1)\cdots(n-q-1)r^n$ is not the conventional basis for this space of sequences, but rather the more easy to write $r^n, nr^n, n^2r^n, \ldots, n^{q-1}r^n$.