1 Recurrence relations, continued continued

Linear homogeneous recurrences are only one of several possible ways to describe a sequence as a recurrence. Here are several other situations which may arise.

1.1 Linear nonhomogeneous recurrence relations

A linear nonhomogeneous recurrence relation is one which has a linear homogeneous form as well as a nonhomogeneous term. here's an example:

Question 1: *How many strings of n numbers from the set $\{0, 1, 2, 3, 4\}$ are there so that there is at least one "1" and the first "1" occurs before any "0"?*

Answer 1: *Let us call the above number a_n, and a string satisfying the above description "valid". To produce a valid string of length n, we can consider every possible first number in the string as an individual case.*

A string beginning with "0" is necessarily invalid. If a string begins with "1", everything that could follow that is valid. Thus, of the 5^{n-1} strings beginning with "1", all are valid. If a string begins with a "2", "3", or "4", then it is valid if and only if the remaining $n-1$ terms form a valid string, which could happen in a_{n-1} ways.

Thus, $a_n = 3a_{n-1} + 5^{n-1}$. This is quite similar to a first-order linear homogeneous differential equation, except it has that inconvenient 5^{n-1} term. Unfortunately, we don't have the tools to solve that, yet!

In addition, note the initial condition $a_0 = 0$, since the string of length zero contains no "1"s and is thus invalid. We could, if we choose, work out a few small values from this: $a_1 = 1$, $a_2 = 8$, $a_3 = 49$, and so forth.

We'll define the class of problems to which this belongs, and discuss solution techniques:

Definition 1. A sequence $\{a_n\}$ is given by a *linear nonhomogeneous recurrence relation of order k* if $a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + \cdots + c_ka_{n-k} + p(n)$ for all $n \geq k$. The recurrence relation $b_n = c_1b_{n-1} + c_2b_{n-2} + c_3b_{n-3} + \cdots + c_kb_{n-k}$ is referred to as the *associated linear homogeneous recurrence relation*

One result is as easy to show for LNRRs as for LHRRs; the following can be proven as a very slight variation of the similar proof for LHRRs.

Proposition 1. *A sequence is uniquely determined by an LNRR of order k and the initial values $a_0, a_1, a_2, \ldots, a_{k-1}$.*

However, there is one very important difference between LNRRs and LHRRs: linear combinations of LNRR-satisfying sequences do no, in general, satisfy the LNRR. Howver, we do have the result:

Proposition 2. *If $\{a_n\}$ satisfies an LNRR, and $\{b_n\}$ satisfies the associated LHRR, then $\{a_n+b_n\}$ satisfies the LNRR.*

Proof. We know that

$$a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + \cdots + c_ka_{n-k} + p(n)$$

and

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + c_3 b_{n-3} + \cdots + c_k b_{n-k}$$

Adding these two equations will give

$$(a_n + b_n) = c_1(a_{n-1} + b_{n-1}) + c_2(a_{n-2} + b_{n-2}) + \cdots + c_k(a_{n-k} + b_{n-k}) + p(n)$$

and thus $\{a_n + b_n\}$ satisfies the LNRR. □

This means that if we have a specific LNRR solution, then we can get a wide range of LNRR solutions simply by adding the associated LHRR solution. Note that this is *very* similar to the method used to solve nonhomogeneous linear differential equations.

The tricky part of this is, of course, coming up with a solution to the LNRR in the first place. Let's try doing that for the example above, where $a_n = 3a_{n-1} + 5^{n-1}$. We might do this by inspired guesswork: since the inhomogeneous term is 5^{n-1}, we mgiht think some multiple of 5^n will do the trick, so suppoce $a_n = C5^n$. Then, the recurrence relation gives us

$$C5^n = 3C \cdot 5^{n-1} + 5^{n-1} = (3C + 1)5^{n-1}$$

so $C = \frac{1}{2}$, and we have the solution $a_n = \frac{1}{2}5^n$. This is a particular solution, and doesn't actually answer the question originally asked (note that in our example, we had the initial condition $a_0 = 0$, not satisfied here).

Now we solve the associated LHRR $b_n = 3b_{n-1}$. Using any of our methods (characteristic polynomial, OGF, or EGF), we'd find the general solution $b_n = k3^n$. Thus, the general solution to our LNRR is $a_n + b_n = \frac{1}{2}5^n + k3^n$. We want a solution which is zero when $n = 0$, so we want $\frac{1}{2}5^0 + k3^0 = 0$, which gives $k = \frac{-1}{2}$. Thus, the solution to our original recurrence relation $a_n = 3a_{n-1} + 4^{n-1}$ with $a_0 = 0$ is $a_n = \frac{5^n - 3^n}{2}$.

The solution method for solving an LNRR with initial conditions, then, is a very minor variation on the LHRR solution method. Given a LNRR $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k} + p(n)$ with initial conditions $a_0, \ldots, a_{k-1}$, this is our process:

1. Find a single sequence $\{a_n^P\}$ to the LNRR.
2. Find the general solution $\{b_n\}$ to the associated LHRR. By the nature of its construction, $\{b_n\}$ will have k undetermined constants.
3. The general solution to the LNRR will be $\{a_n\} = \{a_n^P + b_n\}$. Like $\{b_n\}$, the sequence $\{a_n\}$ will have k undetermined constants in its expression.
4. Setting the known values of $a_0, a_1, \ldots, a_k$ equal to the general-form expressions will yield k equations in k unknowns. Solve for the unknowns to determine the constants in the formula for $\{a_n\}$.

Note that, except for the first step, these are all familiar. The first step is the tricky one. How do we choose the particular solution $\{a_n^P\}$? For now, we rely on inspired guesswork, presuming that if $p(n)$ takes on a particular form, then a_n^P ought to look similar.

If $p(n)$ is a polynomial, we let a_n^P be a polynomial of the same degree, with undetermined coefficients; if $p(n)$ is an exponential, we let a_n^P be an exponential of the same base with an undetermined coefficient, and so forth. Consider, for example, the following LNRR:

Question 2: *What is the closed form for a_n satisfying the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 2n^2$, with $a_0 = 0$ and $a_1 = 1$?*

Answer 2: *Let us start by constructing a single potential solution for the LNRR: $a_n^P = Cn^2 + Dn + E$. Plugging this into the LNRR itself gives:*

$$Cn^2 + Dn + E = 5\left[C(n-1)^2 + D(n-1) + E\right] - 6\left[C(n-2)^2 + D(n-2) + E\right] + 2n^2$$

On collecting all terms on the left side, we see that

$$(2C-2)n^2 + (2D-14C)n + (2E-7D+19C) = 0$$

Since this must be equivalent to zero regardless of n, it follows that $2C - 2 = 0$, $8D - 14C = 0$, and $2E - 7D + 19C = 0$. Thus $C = 1$, $D = 7$, and $E = 15$, which gives the particular solution $a_n^P = n^2 + 7n + 15$.

Now, let us solve the LHRR $b_n = 5b_{n-1} - 6b_{n-2}$. Using methods discovered previously, it is easy to find that the general solution is $b_n = k2^n + \ell 3^n$. Thus, the general solution to the LNRR is $a_n = n^2 + 7n + 15 + k2^n + \ell 3^n$.

Finally, we plug in the initial values:

$$\begin{aligned} 0 = a_0 &= 0^2 + 7\cdot 0 + 15 + k2^0 + \ell 3^0 = 15 + k + \ell \\ 1 = a_1 &= 1^2 + 7\cdot 1 + 15 + k2^1 + \ell 3^1 = 23 + 2k + 3\ell \end{aligned}$$

which we can solve to get $k = -23$, $\ell = 8$, so the solution to the original LNRR with initial conditions is: $a_n = n^2 + 7n + 15 - 23(2^n) + 8(3^n)$.

The choice of $\{a_n^P\}$ is a bit mysterious; we can try to demystify it by approaching this problem with, for instance, an ordinary generating function. Let's follow the lead of the approach we used for homogeneous recurrence relations, letting $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\begin{aligned} \sum_{n=0}^{\infty} a_{n+2}x^n &= \sum_{n=0}^{\infty} \left(5a_{n+1} - 6a_n + 2(n+2)^2\right) x^n \\ \sum_{n=0}^{\infty} a_{n+2}x^{n+2} &= 5x\sum_{n=0}^{\infty} a_{n+1}x^{n+1} - 6x^2\sum_{n=0}^{\infty} a_n + 2\sum_{n=0}^{\infty}(n+2)^2x^{n+2} \\ f(x) - a_1x - a_0 &= 5x(f(x) - a_0) - 6x^2 f(x) + 2\sum_{n=2}^{\infty} n^2x^n \\ (1 - 5x + 6x^2)f(x) &= 5a_0x + a_1x + a_0 + 2\sum_{n=0}^{\infty} n^2x^n - 2\cdot 1^2x - 2\cdot 0 \\ f(x) &= \frac{2\sum_{n=0}^{\infty} n^2x^n - x}{1 - 5x + 6x^2} \end{aligned}$$

so the problem becomes finding an expression for $2\sum_{n=0}^{\infty} n^2x^n$. We have to be pretty clever to do this well:

$$2\sum_{n=0}^{\infty} n^2x^n = \sum_{n=0}^{\infty}\left[A\binom{n+2}{2} + B\binom{n+1}{1} + C\right]x^n$$

which can be solved to find that $A = 4$, $B = -6$, $C = 2$. Then the right side is pretty easy to work out to be $\frac{4}{(1-x)^3} - \frac{6}{(1-x)^2} + \frac{2}{1-x}$. and then we have:

$$f(x) = \frac{4 - 6(1-x) + 2(1-x)^2 - x}{(1-2x)(1-3x)(1-x)^3} = \frac{2x^2 + x}{(1-2x)(1-3x)(1-x)^3}$$

which has partial sum decomposition:

$$f(x) = \frac{3}{2(1-x)^3} + \frac{11}{4(1-x)^2} + \frac{49}{8(1-x)} - \frac{16}{(1-2x)} + \frac{45}{8(1-3x)}$$

which we could express as a sum. Notice that this is far more work than doing it directly would be, but it is possible to go through this procedure without the guesswork of choosing a_n^P, and it in large part justifies that guesswork.

So we can get a library of a_n^P by having a clue of what the ordinary generating functions' denominators look like (note that the numerators are not terribly significant for building our template.

Using the same technique as above, and as was done for the homogeneous case, we can easily work from a generic LNRR to its associated generating function:

Proposition 3. *If a_n satisfies the linear nonhomogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + p(n)$, then it has ordinary generating function*

$$\sum_{n=0}^{\infty} a_n x^n = \frac{A(x) + \sum_{n=0}^{\infty} p(n) x^n}{1 - c_1 x - c_2 x^2 - \cdots - c_k x^k}$$

So the question is, how is $\sum_{n=0}^{\infty} p(n) x^n$ expressed as a rational function, and what appears in its denominator? Whatever appears in its denominator will motivate the a_n^P terms in our solution.

The results here will largely mirror our intuition that a_n^P should "look like" $p(n)$. Let us consider several situations:

When $p(n)$ is an rth-degree polynomial: As seen in the previous example, an rth degree polynomial in n can be expressed as a linear combination of the terms $\binom{n+r}{r}, \binom{n+r-1}{r-1}, \binom{n+r-2}{r-2}, \cdots, \binom{n+2}{2}, \binom{n+1}{1}, \binom{n}{0}$. Since $\sum_{n=0}^{\infty} \binom{n+i}{i} x^n = \frac{1}{(1-x)^i}$, it follows that $\sum_{n=0}^{\infty} p(n) x^n$ is a linear combination of $\frac{1}{(1-x)^{r+1}}, \frac{1}{(1-x)^r}, \frac{1}{(1-x)^{r-1}}, \cdots, \frac{}{(1}$ so

$$\sum_{n=0}^{\infty} p(n) x^n = \frac{Q(x)}{(1-x)^{r+1}}$$

where $Q(x)$ is some (probably awful) polynomial.

When $p(n) = k^n$: Here, $\sum_{n=0}^{\infty} p(n) x^n = \sum_{n=0}^{\infty} (kx)^n = \frac{1}{1-kx}$.

When $p(n) = k^n q(n)$, for some polynomial q of degree r: Here, $\sum_{n=0}^{\infty} p(n) x^n = \sum_{n=0}^{\infty} q(n)(kx)^n$. Using the same techniques as in the polynomial case, we'll find that this is a linear combination of terms of the form $\sum_{n=0}^{\infty} \binom{n+i}{i} (kx)^n = \frac{1}{(1-kx)^i}$, so

$$\sum_{n=0}^{\infty} p(n) x^n = \frac{Q(x)}{(1-kx)^{r+1}}$$

Thus, in general, we end up using the following correspondences:

$p(n)$	OGF denominator	Partial fraction terms	Terms in a_n^P
$a_r n^r + \cdots + a_1 n + a_0$	$(1-x)^{r+1}$	$\frac{A_1}{1-x}, \frac{A_2}{(1-x)^2}, \ldots, \frac{A_{r+1}}{(1-x)^{r+1}}$	$B_r n^r + \cdots + B_1 n + B_0$
ak^n	$1-kx$	$\frac{A}{1-kx}$	Bk^n
$a_r k^n x^r + \cdots + a_1 k^n x + a_0 k^n$	$(1-kx)^{r+1}$	$\frac{A_1}{1-kx}, \frac{A_2}{(1-kx)^2}, \ldots, \frac{A_{r+1}}{(1-kx)^{r+1}}$	$B_r k^n n^r + \cdots + B_1 k^n n + B_0 k^n$

so in general, a_n^P does look a lot like $p(n)$.

There are dangers, though, when one of these terms repeats terms already appearing in the homogeneous solution.

Question 3: *Find a solution to the linear nonhomogeneous recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + n2^n$ with initial conditions $a_0 = 1$, $a_1 = 2$.*

Answer 3: *Slavish adherence to the formula suggests we should try $a_n^P = An2^n + B2^n$. Then, plugging that in, we get:*

$$\begin{aligned} An2^n + B2^n &= 4(A(n-1)2^{n-1} + B2^{n-1}) - 4(A(n-2)2^{n-2} + B2^{n-2}) + n2^n \\ An2^n + B2^n &= 2An2^n - 2A2^n + 2B2^n - An2^n + 2A2^n - B2^n + n2^n \\ An2^n + B2^n &= An2^n + B2^n + n2^n \\ 0 &= n2^n \end{aligned}$$

What just happened? We got a contradiction, suggesting our choice of a_n^P was inadequate. Let's see what would happen in the OGF realm: using our known form for a recurrence relation's OGF, we get:

$$\sum_{n=0}^{\infty} a_n x^n = \frac{P(x) + \sum_{n=0}^{\infty} n2^n x^n}{1-4x-4x^2} = \frac{P(x) + \frac{Q(x)}{(1-2x)^2}}{1-4x-4x^2} = \frac{R(x)}{(1-2x)^4}$$

We'd expect this to lead to solutions of the form $A2^n n^3 + B2^n n^2 + C2^n n + D2^n$. The first two terms are the general solution to the associated LHRR, so the final two terms will be the LNRR-specific one, so here we'd have to use $a_n^P = A2^n n^3 + B2^n n^2$. Plugging that in, we find that

$$A2^n n^3 + B2^n n^2 = 4[A2^{n-1}(n-1)^3 + B2^{n-1}(n-1)^2] - 4[A2^{n-2}(n-2)^3 + B2^{n-2}(n-2)^2] + 2^n n$$

which has solution $A = \frac{1}{6}$, $B = \frac{1}{2}$, so $a_n^P = \frac{2^n n^3}{6} + \frac{2^n n^2}{2}$, which leads to general solution $a_n = \frac{2^n n^3}{6} + \frac{2^n n^2}{2} + k2^n + \ell 2^n n$. Plugging in $a_0 = 1$ gives $k = 1$, and $a_1 = 2$ gives $\ell = \frac{-2}{3}$, so the solution is $a_n = \frac{2^n n^3}{6} + \frac{2^n n^2}{2} + 2^n - \frac{2}{3}2^n n$.

So in general, the following strategy applied for choosing a_n^p:

1. Let a_n^P match $p(n)$ in type: if $p(n)$ is a quadratic, so is a_n^P; if $p(n)$ is an exponential function, so is a_n^P, and so forth. Place undetermined coefficients in front of every term of a_n^P.
2. Calculate the general solution to the associated homogeneous recurrence b_n.
3. If any terms in a_n^P appear in b_n, multiply all terms of a_n^P by n. Repeat until there is no overlap.