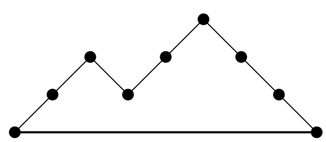
1 Catalan numbers

Previously, we used generating functions to discover the closed form $C_n = 2\binom{1/2}{n+1}(-4)^n$. This will actually turn out to be marvelously simplifiable:

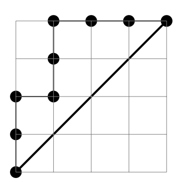
$$\begin{aligned}
C_n &= 2\binom{1/2}{n+1}(-4)^n \\
&= 2(-4)^n \frac{(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2})\cdots(\frac{1-2n}{2})}{(n+1)!} \\
&= (-4)^n \frac{(-1)(-3)(-5)\cdots(1-2n)}{2^n(n+1)!} \\
&= 2^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(n+1)!} \\
&= 2^n \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-1)(2n)}{2\cdot 4\cdot 6\cdots(2n-2)(2n)(n+1)!} \\
&= 2^n \frac{(2n)!}{2^n n!(n+1)!} \\
&= \frac{(2n)!}{n!n!(n+1)} = \frac{1}{n+1}\binom{2n}{n}
\end{aligned}$$

1.1 A purely combinatorial approach

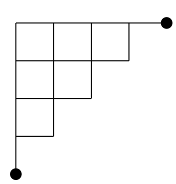
If we'd rather not have this much algebra, take heart: there's a simpler, elegant, combinatorial argument. Let us interpret C_n as the number of paths with n up-steps and n down-steps which don't go below their starting point (such a path is called a *Dyck path*. Let us consider such a path with $n = 4$, drawn with the y-axis representing height and the x-axis representing time:



If we rotate this 45 degrees counterclockwise, and impose a grid structure, we see that this structure corresponds to a combinatorial object much like some we've counted before:



So we might note that C_4 is exactly the number of 8-step walks on the following grid:



We could do this using inclusion-exclusion enumeration of walks through $(1,0)$, $(2,1)$, $(3,2)$ and $(4,3)$, but unfortunately, there are walks going through each of those in every combination, each of them different in number, so such an inclusion-exclusion would be a hideous 16-term monstrosity!

Fortunately, there is a simple way to count these after all, by building a bijection between walks to (n,n) which touch the subdiagonal and walks to $(n+1,n-1)$.

Consider a walk from $(0,0)$ to (n,n) which is not always above the subdiagonal. Then, there is at least one point (x,y) on it where $x>y$; consider the first such point, which can easily be seen to have the form $(x,x-1)$. We may consider the walk from $(0,0)$ to (n,n) as being a composition of two walks: one from $(0,0)$ to $(x,x-1)$, and one from $(x,x-1)$ to (n,n). This latter walk consists of $n-x$ right-steps and $n-x+1$ up-steps. Now let us flip this walk, replacing up-steps with right-steps and vice-versa, so now we have a walk from $(x,x-1)$ consisting of $n-x+1$ right-steps and $n-x$ up-steps; this will be a walk to $(n+1,n-1)$. Thus, we may map each walk from $(0,0)$ to (n,n) which goes below the diagonal to a walk from $(0,0)$ to $(h+1,n-1)$.

To demonstrate that this is a bijection, let us observe how this process could be reversed. A walk from $(0,0)$ to $(n+1,n-1)$ must necessarily pass through a point (x,y) where $x>y$: the destination is such a point, even if no other point is. Consider the first such point in the walk; as above, its coordinates must be $(x,x-1)$, and now we consider this walk as a composition of a walk from $(0,0)$ to $(x,x-1)$ and from $(x,x-1)$ to $(n+1,n-1)$. The latter walk consists of $n-x+1$ right-steps and $n-x$ up-steps; flipping it gives us a walk from $(0,0)$ to (n,n) passing through $(x,x-1)$.

Thus, to find the number of walks which do not dip below the diagonal in an $n\times n$ grid, we take the number of walks in an $n\times n$ grid, and subtract the nubmer of walks in an $(n+1)\times(n-1)$ grid. This is easily calculated to be

$$\binom{2n}{n}-\binom{2n}{n-1}=\frac{2n!}{n!n!}-\frac{2n!}{(n+1)!(n-1)!}=\frac{2n!}{n!n!}\left(1-\frac{n}{n+1}\right)=\frac{1}{n+1}\binom{2n}{n}$$

2 Stirling's approximation

Counting the number of ways something can happen has some bearing on computational complexity. If a computer was, for instance, instructed to find the pair of numbers with least difference in a set on n nubmers, one way would be to work by brute force, testing every pair of numbers, which would involve $\binom{n}{2}$ individual tests: this algorithm would take $\frac{n(n-1)}{2}=\frac{n^2-n}{2}$ steps, which would be called, in algorithmic circles, simply a "$O(n^2)$" algorithm to represent the asymptotically largest term in the number of steps, without a coefficient. So having an idea of the rough asymptotic value of, for instance, a binomial coefficient, can be quite useful.

We can see easily that $\binom{n}{2}\approx\frac{n^2}{2}=O(n^2)$ and that $\binom{n}{3}\approx\frac{n^3}{6}=O(n^3)$, but we are on shakier ground trying to say that in general $\binom{n}{k}\approx\frac{n^k}{k!}=O(n^k)$, since if k is large enough, the terms in the numerator of $\frac{n(n-1)\cdots(n-k+1)}{k!}$ are not actually all that close to n.

A tool which serves us well in approximating binomials (among other things) is called *Stirling's approximation.*

Theorem 1. *For sufficiently large n, $n!\approx\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$; more precisely, it is true that*

$$\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}=1$$

This formula is traditionally proven using analytical methods, sometimes invoking the definition of the gamma function:

$$n! = \Gamma(n+1) = \int_0^\infty t^n e^{-t} dt$$

The analytical methods used to show that this is equal to $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot \Psi(n)$, where $\Psi(n)$ tends towards 1 for large n, are rather beyond the scope of this course; a decent proof appears in *Rudin's Principles of Mathematical Analysis*.

However, our main use for Stirling's approximation will be in giving alternative, factorial-free forms for our favorite enumerations. For instance, we can determine, asymptotically, what $\binom{n}{k}$ is roughly equal to, in terms of n and k:

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{(n-k)!k!} \\
&\approx \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi(n-k)}\left(\frac{n-k}{e}\right)^{n-k}\sqrt{2\pi k}\left(\frac{k}{e}\right)^k} \\
&\approx \sqrt{\frac{n}{2\pi(n-k)k}}\frac{n^n}{(n-k)^{n-k}k^k} \\
&\approx \frac{1}{\sqrt{2\pi}}\frac{n^{n+1/2}}{(n-k)^{n-k+1/2}k^{k+1/2}}
\end{aligned}$$

When $k \ll n$, we can assume $n-k \approx n$ and treat $k^{k+1/2}$ as an arbitrary constant, so the above would become:

$$\binom{n}{k} \approx \frac{1}{\sqrt{2\pi}k^{k+1/2}}\frac{n^{n+1/2}}{n^{n-k+1/2}} = \frac{n^k}{\sqrt{2\pi}k^{k+1/2}}$$

so that if k is a constant small with respect to n, we can justifiably claim that $\binom{n}{k} = O(n^k)$. But if k is a significant proportion of n, for instance, if $k = cn$ for some constant $0 < c < 1$:

$$\binom{n}{cn} \approx \frac{1}{\sqrt{2\pi}}\frac{n^{n+1/2}}{((1-c)n)^{(1-c)n+1/2}(cn)^{cn+1/2}} \approx \frac{1}{\sqrt{2\pi nc(1-c)}\left[(1-c)^{1-c}c^c\right]^n}$$

which would asymptotically be $O(\frac{a^n}{\sqrt{n}})$, where $a = \frac{1}{(1-c)^{1-c}c^c}$.

Turning to a specific useful example where the terms of binomials are constant multiples of each other, we may consider deriving an approximation for the above-determined formula for the Catalan numbers:

$$\begin{aligned}
C_n &= \frac{1}{n+1}\binom{2n}{n} \\
&\approx \frac{1}{(n+1)\sqrt{2\pi}}\frac{(2n)^{2n+1/2}}{n^{n+1/2}n^{n+1/2}} \\
&\approx \frac{1}{(n+1)\sqrt{2\pi}}\frac{2^{2n+1/2}}{n^{1/2}} \\
&\approx \frac{4^n}{(n+1)\sqrt{\pi n}}
\end{aligned}$$

This is a very simple formula, and one that gives an obvious asymptotic of $O(\frac{4^n}{n^{3/2}}$. And it's actually fairly accurate!

n	1	2	3	4	5	6	7	8	9	10
C_n	1	2	5	14	42	132	429	1430	4862	16796
$\frac{4^n}{(n+1)\sqrt{\pi n}}$	1.13	2.13	5.21	14.44	43.06	134.78	436.72	1452.50	4929.96	17007.20

Incidentally, based on our previous formula $C_n = 2\binom{1/2}{n+1}(-4)^n$, we now have a strong suggestion that $\binom{1/2}{n+1} \approx \frac{(-1)^n}{2(n+1)\sqrt{\pi n}}$. This isn't terribly useful on its own, but it's cute.

3 Symmetry and the Polya Method

We flirted with symmetry-reduction way back at the beginning of the course. Let's look at a pair of sample questions illustrating types of symmetry and how we'd deal with it:

Question 1: *We have 7 distinct people from which we'd like to form a bridge table (of 4 people in an order). Rotations of the same foursome are identical. How many foursomes are there?*

Answer 1: *If we just consider an ordered list of 4 people, this is a simple enumeration statistic:* $_7P_4 = \frac{7!}{3!} = 840$. *But this includes the same foursomes multiple times: note that ABCD is the same as DABC, CDAB, and BCDA (but not, for instance, ADCB or BACD, which are not simple rotations). Fortunately, every foursome will be represented* exactly *four times, since it and its 3 rotations are guaranteed to appear in our list of 840. Thus, there are* $\frac{840}{4} = 210$ *bridge foursomes.*

Question 2: *We have 7 individual distinct beads, and would like to put 4 on a necklace. Rotations and flips of a single necklace are considered to be identical. How many different necklaces are possible?*

Answer 2: *This is much like above, but here, instead of having only 4 necklaces equivalent to ABCD, we have 8: ABCD itself, DABC, CDAB, BCDA, DCBA, ADCB, BADC, and CBAD. Thus there are* $\frac{840}{8} = 105$ *necklaces.*

These are simple ways of winnowing down symmetries. Let's think of a more difficult case!

Question 3: *We have an unlimited supply of black and white beads. We want to make a 5-bead necklace. How many different ways are there to do this?*

Answer 3: *This is not easily done with pure symmetry division. We might think that there are 10 transformations, so each necklace is represented 10 times if we were to just consider straight strings, but there would be* $2^5 = 32$ *strings, and* $\frac{2^5}{10} = 3.2$ *is, first of all, not an integer, and second, too small.*

If we were to brute-force this, we'll find 8 necklaces (all-white, one black, two adjacent blacks, two non-adjacent blacks, two adjacent whites, two non-adjacent whites, one white, all-black). How could we possibly get this result in a sensible manner?

To do this, we're going to need to discuss equivalence classes and permutation groups.

3.1 Permutation groups

We'ev encountered permutations as combinatorial objects, but here we're going to want to discuss their algebraic properties.

Definition 1. A *permutation of length n* is a bijective function $\pi : \{1, 2, 3, \ldots, n\} \to \{1, 2, 3, \ldots, n\}$. The *product* $\pi\sigma$ of two permutations π and σ is their composition as functions, $\pi \circ \sigma$. The *inverse* π^{-1} of a permutation π is its inverse as a function.

Note that permutation products are not commutative in general: $\pi\sigma$ is likely to be a different permutation than $\sigma\pi$. In addition, for e being the identity permutation (i.e. the identity function on $\{1, 2, 3, \ldots, n\}$, it is easy to see that $\pi e = e\pi = \pi$, and that $\pi\pi^{-1} = \pi^{-1}\pi = e$. Together with the associative law of multiplication, these properties guarantee that the permutations form the algebraic structure known as a group; this group is known as S_n, the symmetric group, and we know, from our combinatorial explorations, that $|S_n| = n!$.

There are two standards for representing a permutation: there is "mapping" notation, in which $\pi(1)$, $\pi(2)$, $\pi(3)$, etc. are listed in order; there is also cycle notation, in which one lists each element followed by its image, repeating until the cycle returns to the beginning. For instance, considering an element of C_6 with $\pi(1) = 3$, $\pi(2) = 2$, $\pi(3) = 5$, $\pi(4) = 6$, $\pi(5) = 1$, and $\pi(6) = 4$, this permutation could be represented in "mapping" notation as (325614) and in cycle notation as (135)(2)(46). In these notes, mapping notation is used for brevity.

Any subgroup of S_n is known as a *symmetry group*. From an algebraically naïve perspective, we can describe a symmetry group as such:

Definition 2. A *symmetry group* G is a nonempty set of permutations of the same length such that:

1. the identity is in G.
2. for σ and π in G, $\sigma\pi$ is in G.
3. for π in G, π^{-1} is in G.

As it turns out, only the second of these conditions is actually necessary (the first follows since for any π in G, there is an N such that $\pi^N = e$, and the third follows by noting that $\pi^{N-1} = \pi^{-1}$), but the first and third give a better idea of its structure.

There are a tremendous variety of symmetry groups, and the results we will see later apply to all of them. For now, though, we will focus on those with a geometric representation.

3.2 2-dimensional symmetries

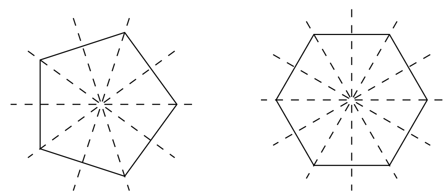
Symmetries in the 2-dimensional plane are mappings of a regular n-gon onto itself. We can envision two such symmetries: rotational only (such as in our bridge game example) or rotation with reflection (as in our necklance example).

If only rotations are allowed, we can map an n-gon to itself with a rotation of $\frac{2\pi k}{n}$ radians; that is to say, with any multiple of $\frac{1}{n}$ revolutions. This yields n different possible permutations: the identity rotation of 0 radians, a rotation of $\frac{2\pi}{n}$ radians, a rotation of $\frac{4\pi}{n}$ radians, and so forth up to $\frac{2\pi(n-1)}{n}$ radians. The next rotation in this sequence would be identical to the identity, and thus has already been counted.

Algebraically, these can be represented as forward-shifts of the vertices: if we label the vertices of an n-gon cyclically, then these rotations would respectively be $e = (123 \cdots n)$, $r = (234 \cdots n1)$, $r^2 = (34 \cdots n12)$, and so forth up to $r^{n-1} = (n123 \cdots (n-2)(n-1))$.

This group is associated with the *cyclic group*, denoted C_n or $\mathbb{Z}_n$, and can be thought of in terms of modular arithmetic: $r^a + r^b = r^c$, where $c \equiv a + b \pmod{n}$.

If we allow reflection, then we get a larger group, which includes all the rotations, and in addition, includes n other permutations. If we identify these n other permutations algebraically, we can define $f = (n(n-1)(n-2)\cdots 321)$ and then note that a group containing f and r must also contain $fr = ((n-1)(n-2)\cdots 321n)$, $fr^2 = ((n-2)(n-3)\cdots 21n(n-1))$ and so forth up to $fr^{n-1} = (1n(n-1)\cdots 432)$. Alternatively, geometrically, we can note that a composition of reflections with rotations is itself a reflection, and that there are n rotations mapping an n-gon onto itself. The axes of rotation look somewhat different depending on whether n is even or odd, but there are always n of them:



This permutation group is known as the *dihedral* group D_n. $|D_n| = 2n$, and conventionally, elements of D_n are written as fr^k with $0 \leq k < n$. The product of two elements is determined according to the rules $r^n = 1$, $f^2 = 1$, and $fr = r^{-1}f$.

3.3 3-dimensional symmetries

Symmetry in 3-space is generally limited to the 5 platonic solids, which means that there are 3 figures in 3-space whose symmetries are frequently explored: the tetrahedron, the cube (which has the same symmetries as the octahedron, which is dual to the cube), and the icosohedron (which has the same symmetries as the dodecahedron)

Rotational axes which map a solid to itself must pass through opposite features of the solid: either through the centers of a pair of opposite faces, a pair of opposite vertices, the midpoints of a pair of opposite edges, or the center of a face opposite a vertex. This allows us to enumerate the rotational axes of any solid with comparative ease.

We may start by considering a tetrahedron. A tetrahedron has 4 vertices and 4 faces, with faces opposite vertices, and 6 edges. We thus have 7 prospective axes of symmetry: 4 drawn between a vertex and its opposite face, and 3 drawn between opposite pairs of edges.

Now we may use these axes of symmetry to identify all the permutations corresponding to rotations of the tetrahedron. There is the identity permutation, which corresponds to a zero rotation. For each of the vertex-to-face axes, a view of the tetrahedron from above has triangular symmetry, so we could rotate around these axes either 120° or 240°; we have 4 such axes with 2 possible rotations, giving eight permutations. Lastly, for each of the three edge-to-edge axes, we may rotate the edge onto itself via an 180° rotation, giving three more permutations. Thus, there are $1 + 8 + 3 = 12$ rotations mapping the tetrahedron onto itself. This group is sometimes called the *chiral tetrahedral symmetry group*, but is also associated with the algebraic group called the *alternating group of degree 4*, or A_4.

If we add reflections to this group, or rather, add a single reflection and count on composition with rotations to yield other reflections, then we double the group's size, getting 24 automorphisms of the tetrahedron. This is the *achiral tetrahedral symmetry group*, algebraically associated with the group of all permutations on 4 elements, the symmetric group S_4.

Moving on we may look at the cube: it has 6 faces divided into 3 opposite pairs, 12 edges divided into 6 opposite pairs, and 8 vertices divided into 4 opposite pairs. Thus there are 13 possible axes of rotation through the cube. Looknig directly at a face, the cube has 4-fold symmetry. Thus, the face-to-face axes can be rotated 90°, 180°, or 270° to map the cube back onto itself. Since there are 3 such axes and 3 possible rotations about each, these yield 9 permutations of the cube. The edge-to-edge axes can only map the cube to itself by going through an 180° rotation, so these 6 axes contribute one permutation each for a total of 6. Lastly, viewing a corner of the cube, the cube has threefold symmetry, so a vertex-to-vertex axis can be rotated either 120° or 240°; we have 4 such axes with 2 possible rotations, giving 8 permutations. Together with the identity permutation, the above total $9 + 6 + 8 + 1 = 24$ in number. This group is the *chiral octahedral symmetry group*, denoted O, and sometimes considered as a subgroup of S_6. It is in fact isomorphic to the achiral tetrahedral symmetry group, although it is rarely considered in that context.

As above, adding a reflection doubles the order of the group, so the automorphisms of the cube by rotation *and* reflection number 48; this group is called the *achiral octahedral symmetry group* or simply the *octahedral group* and is denoted O_h.