1 Posets

1.1 Extreme elements

Last week we defined maximal, minimal, greatest, and least elements of a poset. We will explicitly determine the useful properties of these elements.

The proofs below will generally only present an argument for maximal and greatest elements; the same argument can be modified very slightly to work for minimal and least elements.

Proposition 1. *If $(S, \preceq)$ is a nonempty finite poset, then S has at least one maximal element (and at least one minimal element).*

Proof. Let $x_0 \in S$. If x_0 is maximal, than S clearly has a maximal element; otherwise, there must be an $x_1 \neq x_0$ such that $x_0 \preceq x_1$. Now let us subject x_1 to the same consideration; if x_1 is maximal, we are done, otherwise there is an $x_2 \neq x_1$ such that $x_1 \preceq x_2$. Iterating this process must necessarily have one of two results: either there will be some x_i which is maximal, or the sequence $x_0 \preceq x_1 \preceq x_2 \preceq x_3 \preceq \cdots$ is infinite. If some element is maximal, we will achieve our goal; we shall show that the second situation is impossible.

Suppose we do have such an infinite sequence of elements of S, so that each $x_i \preceq x_{i+1}$ and $x_i \neq x_{i+1}$. Then, since this sequence is infinite and its elements are drawn from the finite set S, the same element must appear twice in the sequence (a massive understatement, but all we need). Thus we may assume there are $i < j$ such that $x_i = x_j$. By transitivity, since $x_i \preceq x_{i+1} \preceq \cdots \preceq x_{j-1}$, it follows that $x_i \preceq x_{j-1}$. However, since $x_i = x_j$ and $x_{j-1} \preceq x_j$, it is also true that $x_{j-1} \preceq x_i$. This can only be true, by antisymmetry, if $x_i = x_{j-1}$. However, we also know that $x_{j-1} \neq x_j = x_i$, leading to a contradiction. □

Note that this is emphatically not true for infinite sets S; they may have maximal elements, as $([0, 1], \leq)$ does, or need not, as exhibited by $(\mathbb{Z}^+, \leq)$.

Proposition 2. *If a poset $(S, \preceq)$ has a greatest element (or a least element) then it has exactly one maximal (minimal) element. Furthermore, its greatest (least) element is exactly the unique maximal (minimal) element.*

Proof. First, we shall prove that a greatest element must be maximal. This is easy: suppose x is nonmaximal: then there is some $y \neq x$ such that $x \preceq y$. By antisymmetry, since x and y are nonequal, it is clear that $y \not\preceq x$. Thus, since there is an element y which x is not greater than, x is not a greatest element.

Now, let us show that a poset with two or more maximal elements cannot have a greatest element. Suppose x is the greatest element of a poset S with at least two maximal elements. Since there are two or more maximal elements, there is at least one maximal element not equal to x, which we shall call y. Since $x \neq y$ and y is maximal, $y \not\preceq x$, which contradicts our presumption that x is greater than every element of S. □

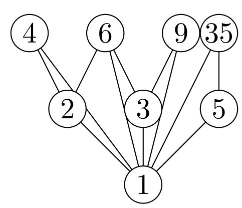
Corollary 1. *A* finite *poset $(S, \preceq)$ has a greatest element (or a least element)* if and only if *it has exactly one maximal (minimal) element. Furthermore, its greatest (least) element is exactly the unique maximal (minimal) element.*

Proof. One direction follows from the proposition above; all we need to show now is that if there is a unique maximal element of a poset, it is in fact the greatest element. Suppose x is the unique maximum of S. If x is not the greatest element of S, then there is some y_0 such that $y_0 \not\preceq x$. By maximality of x, $x \not\preceq y_0$, so x and y_0 are incomparable. Since y_0 is not maximal, there is a $y_1 \neq y_0$ such that $y_0 \preceq y_1$. If $y_1 \preceq x$, then $y_0 \preceq x$ by transitivity, so $y_1 \not\preceq x$. However, by maximality of y_1, $x \not\preceq y_1$, so x and y_1 are incomparable. We may proceed with this sequence, producing a sequence $y_0 \preceq y_1 \preceq y_2 \preceq \cdots$, which cannot terminate since all the y_i are distinct from x and thus nonmaximal. We saw previously that such a sequence is impossible, so such a y_0 does not exist, and thus x is in fact the greatest element of S. □

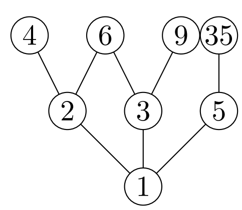
Note that the above may again be untrue when S is infinite. Consider $S = \mathbb{Z} \cup \{\mu\}$, with the ordering $\preceq$ defined by $a \preceq b$ when $a \leq b \in \mathbb{Z}$, and $a \not\preceq b$ if either $a = \mu$ or $b = \mu$. Then μ is incomparable to everything and thus the set's unique maximal element, but *not* a greatest element.

1.2 Visualization with Hasse diagrams

We can draw a poset (preferably) by placing x below y if $x \preceq y$ and drawing an edge from x to y. This can get pretty messy if we tried to do it on a moderately large poset. For instance, if we discussed the divisibility relation on $\{1, 2, 3, 4, 5, 6, 9, 35\}$, we get the moderately ugly diagram:



This has a lot of redundant lines on it. If instead we only connect a to b when $a \preceq b$ and there is no c not such that $a \preceq c \preceq b$, we get the cleaner diagram:



On a Hasse diagram, we can come up with easy visual interpretations for every aspect of a poset: maximal elements are ones without upwards-facing branches; minimal elements are those without downwards-facing branches, a chain is a collection of vertices in linear order.

Our explorations from here will be into chains and antichains: finding maximal chains and antichains, and the significance thereof.