1. **(15 points)** *Find the following derivatives:*

   (a) **(5 points)** *For* $y = \frac{x^7 - x^3}{2+x}$, *evaluate* $\frac{dy}{dx}$.
   Using the quotient rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\frac{x^7 - x^3}{2+x} \\ &= \frac{(2+x)\frac{d}{dx}(x^7 - x^3) - (x^7 - x^3)\frac{d}{dx}(2+x)}{(2+x)^2} \\ &= \frac{(2+x)(7x^6 - 3x^2) - (x^7 - x^3)(1)}{(2+x)^2}\end{aligned}$$

   which need not be further simplified.

   (b) **(5 points)** *For* $f(x) = (x^5 - 3\sqrt{x})(2x^3 + 4x - 7)$, *evaluate* $f'(x)$.
   Using the product rule,

$$\begin{aligned}f(x) &= \frac{d}{dx}\left[(x^5 - 3\sqrt{x})(2x^3 + 4x - 7)\right] \\ &= \left[\frac{d}{dx}(x^5 - 3x^{1/2})\right](2x^3 + 4x - 7) + (x^5 - 3\sqrt{x})\frac{d}{dx}(2x^3 + 4x - 7) \\ &= (x^5 - \frac{3}{2}x^{-1/2})(2x^3 + 4x - 7) + (x^5 - 3\sqrt{x})(6x^2 + 4)\end{aligned}$$

   which need not be further simplified.

   (c) **(5 points)** *Evaluate* $\frac{d}{dt}\left(4t - 3 + \frac{5}{t}\right)^5$.
   This is a composition of two functions we can differentiate: $u = 4t - 3 + \frac{5}{t}$ and $y = u^5$. What we seek is $\frac{dy}{dt}$, which by the chain rule is:

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{du}\frac{du}{dt} \\ &= \left(\frac{d}{du}u^5\right)\frac{d}{dt}(4t - 3 + 5t^{-1}) \\ &= (5u^4)(4 - 5t^{-2}) \\ &= 5\left(4t - 3 + \frac{5}{t}\right)^4(4 - 5t^{-2})\end{aligned}$$

2. **(20 points)** *Perform the following differentiations:*

   (a) **(10 points)** $\frac{d}{dt}\left(\frac{t^2-2}{t^3+4t}(3t^2 + 2t - 4)\right)$.
   This problem will require the product rule and quotient rule applied in sequence:

$$\begin{aligned}\frac{dy}{dt} &= \left(\frac{d}{dt}\frac{t^2 - 2}{t^3 + 4t}\right)(3t^2 + 2t - 4) + \frac{t^2 - 2}{t^3 + 4t}\frac{d}{dt}(3t^2 + 2t - 4) \\ &= \left(\frac{d}{dt}\frac{t^2 - 2}{t^3 + 4t}\right)(3t^2 + 2t - 4) + \frac{t^2 - 2}{t^3 + 4t}(6t + 2) \\ &= \frac{(t^3 + 4t)\frac{d}{dt}(t^2 - 2) - (t^2 - 2)\frac{d}{dt}(t^3 + 4t)}{(t^3 + 4t)^2}(3t^2 + 2t - 4) + \frac{t^2 - 2}{t^3 + 4t}(6t + 2) \\ &= \frac{(t^3 + 4t)(2t) - (t^2 - 2)(3t^2 + 4)}{(t^3 + 4t)^2}(3t^2 + 2t - 4) + \frac{t^2 - 2}{t^3 + 4t}(6t + 2)\end{aligned}$$

which need not be further simplified.

(b) **(10 points)** $\frac{d}{dx}\left(\frac{4x-3}{x^3-2x}\right)^3$.

This problem will require the quotient rule and the chain rule applied in sequence: if we call the entire expression y, the subdivisio0n into easily differentated components is as follows: $y = u^3$, $u = \frac{4x-3}{x^3-2x}$. It is easy to see that $\frac{dy}{du} = 3u^2$, but finding $\frac{du}{dx}$ requires the quotient rule:

$$\begin{aligned}
\frac{du}{dx} &= \frac{d}{dx}\left(\frac{4x-3}{x^3-2x}\right) \\
&= \frac{(x^3-2x)\frac{d}{dx}(4x-3) - (4x-3)\frac{d}{dx}(x^3-2x)}{(x^3-2x)^2} \\
&= \frac{(x^3-2x)(4) - (4x-3)(3x^3-2)}{(x^3-2x)^2}
\end{aligned}$$

And thus:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\
&= (3u^2)\left(\frac{(x^3-2x)(4) - (4x-3)(3x^3-2)}{(x^3-2x)^2}\right) \\
&= 3\left(\frac{4x-3}{x^3-2x}\right)^2\left(\frac{(x^3-2x)(4) - (4x-3)(3x^3-2)}{(x^3-2x)^2}\right)
\end{aligned}$$

which need not be further simplified.

3. **(5 points)** *Use differentials to find a rational approximation of* $\sqrt{3.97}$.

The closest value to 3.97 with a known square root is 4, so we'll approximate based on the known values at 4. Thus, if $f(x) = \sqrt{x}$, we have $f'(x) = \frac{1}{2\sqrt{x}}$ and specifically that $f(4) = \sqrt{4} = 2$, and $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. Our differential approximation is thus

$$f(3.97) \approx f(4) + (-0.03)f'(4) = 2 + (-0.03)\frac{1}{4} = 2 - 0.0075 = 1.9925$$

Note that the actual value of $\sqrt{3.97}$ to 6 decimal places is 1.992486, which this approximates quite well.

4. **(15 points)** *Answer the following questions about the ellipse determined by the equation* $2x^2 - 4xy + 5y^2 - 4x + 3y = 12$.

(a) **(10 points)** *Find an expression for* $\frac{dy}{dx}$ *in terms of* x *and* y.

Differentiating both sides of the expression:

$$\begin{aligned}
\frac{d}{dx}\left(2x^2 - 4xy + 5y^2 - 4x + 3y\right) &= \frac{d}{dx}12 \\
4x - 4\frac{d}{dx}(xy) + 5\frac{d}{dx}y^2 - 4 + 3\frac{dy}{dx} &= 0 \\
4x - 4\left[(\frac{d}{dx}x)y + x\frac{dy}{dx}\right] + 5\frac{dy}{dx}\frac{d}{dy}y^2 - 4 + 3\frac{dy}{dx} &= 0 \\
4x - 4\left[y + x\frac{dy}{dx}\right] + 5\frac{dy}{dx}(2y) - 4 + 3\frac{dy}{dx} &= 0 \\
4x - 4y - 4x\frac{dy}{dx} + 10y\frac{dy}{dx} - 4 + 3\frac{dy}{dx} &= 0 \\
4x\frac{dy}{dx} + 10y\frac{dy}{dx} + 3\frac{dy}{dx} &= 4y - 4x - 4 \\
(4x + 10y + 3)\frac{dy}{dx} &= 4y - 4x - 4 \\
\frac{dy}{dx} &= \frac{4y - 4x + 4}{4x + 10y + 3}
\end{aligned}$$

(b) **(5 points)** *Find the equation of the tangent line to this ellipse at the point* $(-1, -2)$.

Using the formula above, we know that when $x = -1$ and $y = -2$, $\frac{dy}{dx} = \frac{4(-2)-4(-1)+4}{4(-1)+10(-2)+3} = \frac{0}{-21} = 0$ We thus want the equation of a line through $(-1, -2)$ with slope 0. This is the equation $y = 0x + b$, or just $y = b$, which, to pass through $(-1, -2)$, would need to have $b = -2$. Thus the equation of the tangent line is $y = -2$.