1. **(15 points)** *You are constructing an open-topped cardboard box with a square base which must have a volume of thirty-two cubic feet. What is the least amount of cardboard you could use to do so?*

   Call the side-length of the base x, and the height h. Our required volume thus translates to the restriction$x^2h = 32$, while our goal is to minimize the overall surface area given by $x^2 + 4xh$ (the sum of the area of the square base and the four identical lateral faces). Our restriction induces the relationship $h = \frac{32}{x^2}$, so applying this relationship to the cost gives the cost as the one-variable function $f(x) = x^2 + \frac{128}{x}$ Note that x can be any positive number, but that values of x very close to zero or very large induce unreasonably large costs $f(x)$, so our cost-minimizing choice will be an extremum of $f(x)$.

   $f'(x) = 2x - \frac{128}{x^2}$, so $f'(x) = 0$ if $2x = \frac{128}{x^2}$, which occurs when $x = \sqrt[3]{64} = 4$. Then $f(x) = 4^2 + \frac{128}{4} = 48$ and $h = \frac{16}{4^2} = 2$, so our optimal box (with a base-side-length of four feet, and a height of two feet) uses 48 square feet of cardboard.

2. **(15 points)** *Answer the following questions related to the shape of the graph of* $f(x) = x^3 + 6x^2 - 15x + 7$.

   (a) **(3 points)** *Where is it increasing? Where is it decreasing?*

   $f'(x) = 3x^2 + 12x - 15 = 3(x^2 + 4x - 5) = 3(x + 5)(x - 1)$; multiplying constituent parts and noting their sign changes, we see that $f'(x)$ is positive (since both factors are negative) if $x < -5$, negative if $-5 < x < 1$, and positive if $x > 1$. Thus $f(x)$ is increasing when $x < -5$ and $x > 1$, and decreasing when $-5 < x < 1$ (some definitions of increase and decrease include the $f'(x) = 0$ case, so these intervals may include their endpoints, if desired).

   (b) **(3 points)** *What are its critical points, and is each a local maximum, a local minimum, or neither?*

   From the factorization above, it is clear that $f(x) = 0$ when $x = -5$ and $x = 1$. Since $f(x)$ increases up to -5 and decreases from it, it is clearly a local maximum; since it decreases to 1 and increases after, $x = 1$ is a minimum. This result can also be obtained via the second derivative test.

   (c) **(4 points)** *Where is it concave up? Where is it concave down? Does it have any points of inflection?*

   $f''(x) = 6x + 12$, so $f''(x) > 0$ if $x > -2$, and $f''(x) < 0$ if $x < -2$. Thus, $f(x)$ is concave up when $x > -2$, concave down when $x < -2$, and has a point of inflection at $x = -2$.

3. **(15 points)** Transylvania Polygnostic University currently has 3000 students. Enrollment is expected to rise by 4% each year.

   (a) **(3 points)** *Create a function $f(t)$ to describe the expected number of students t years from now.*

   Since enrollment grows by 4% per year, the enrollemnt after a year is 104% of the former enrollment; thus, in t years, the university has a factor equal to 1.04^t multiplied by the original enrollment. Thus, $f(t) = 3000\,(1.04^t)$.

   (b) **(5 points)** *How many years will it take for enrollment to reach 4000 students?*

We solve for t when $f(t) = 4000$:

$$\begin{aligned} 3000\left(0.82^t\right) &= 4000 \\ 1.04^t &= \frac{4000}{3000} = \frac{4}{3} \\ t = \log_{1.04}\frac{4}{3} &= \frac{\ln\frac{4}{3}}{\ln 1.04} \approx 7.3 \text{ years} \end{aligned}$$

The approximation given here would not be part of an answer detemrined without the assistance of a calculator.

4. **(10 points)** Answer the following questions related to the shape of the graph of $g(x) = \frac{x^2+5x+13}{x+1}$

   (a) **(5 points)** *Where is it increasing? Where is it decreasing? Identify its local extrema.*
   Using the quotient rule:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\frac{x^2+5x+13}{x+1} \\ &= \frac{(x+1)\frac{d}{dx}(x^2+5x+13) - (x^2+5x+13)\frac{d}{dx}(x+1)}{(x+1)^2} \\ &= \frac{(x+1)(2x+5) - (x^2+5x+13)(1)}{(x+1)^2} \\ &= \frac{(2x^2+7x+5) - (x^2+5x+13)}{(x+1)^2} \\ &= \frac{x^2+2x-8}{(x+1)^2} = \frac{(x+4)(x-2)}{(x+1)^2} \end{aligned}$$

   The denominator has no effect on the sign, so this expression is positive when $x < -4$ or $x > 2$, and negative for $-4 < x < 2$. Thus, $x = -4$ is a maximum, as a transition from increase to decrease, and $x = -2$ is similarly a minimum. The types of extrema could also be tested with the second derivative test, but doing so in this case is a lot of work to little purpose.

   (b) **(5 points)** *Which of its local extrema are also global extrema, and why?*
   Neither local extremum is a global extremum, for two reasons: first of all, the long-term behavior of the function on the left and right is to become arbitarily high-magnitude megative and positive values respectively, so no particular values can be global extrema. In addition, very close to $x = -1$, the function likewise achieves very high positive and negative magnitudes, so once more no individual point can lay claim to the position of "highest" or "lowest".