1. *Let G be a simple graph with n vertices and m edges. Show that G is a complete graph if and only if $m = \binom{n}{2}$.*

   First, we shall prove that a complete graph on n vertices has $\binom{n}{2}$ edges. In a complete graph every vertex is adjacent to every other vertex; thus, every pair of distinct vertices are joined by one edge (and no more than one edge, since the graph is simple); in addition, simplicity guarantees that no edges exist excpet those joining two distinct vertices. Thus, the edges of G are equinumerous with the pairs of distinct vertices of G. This latter quantity is clearly the number of ways to choose 2 elements of an n-element set, which is $\binom{n}{2}$.

   Conversely, if G is simple, the total number of possible edges between distinct vertices of G is $\binom{n}{2}$ for the reasons given above; thus, if G contains $\binom{n}{2}$ edges, since, no two edges can join the same pair of vertices, there must be an edge joining every pair of distinct vertices, so G is complete.

2. *Find graphs G and H such that H is a subgraph of G, but is not an induced subgraph of G.*

   The simplest example of this is the graph $G = P_2$ with H consisting of two isolated vertices. Many other examples are possible.

3. *If a graph G has n vertices and m edges, is there a maximum length for a walk, path, or trail? If these maximums exist, what are they and why? If they do not exist, why not?*

   A walk can be arbitrarily long, by virtue of repitition: even on as simple a graph as P_2, one could construct an arbitrarily long walk by visiting the two vertices alternatingly for as long as desired.

   A path can visit each vertex no more than once; thus it contains at most n vertices in the list of vertices visited, and traverses $n-1$ edges between them, so a path can be of length at most $n-1$ (a graph *need not* contain a path of length n; the bipartite graph $K_{1,(n-1)}$ has no path of length greater than 2, for instance, no matter how large n is).

   A trail, likewise, can traverse each edge no more than once, so it traverses no more than m edges, so it has maximum length of m (as above, this bound may not be attained for certain graphs).

4. *If a graph G has n vertices and m edges, prove that the minimum vertex degree δ and maximum vertex degree Δ in the graph are subject to the equation $\delta \leq \frac{2m}{n} \leq \Delta$.*

   We know that $\sum_{v \in V(G)} d(v) = 2\|G\| = 2m$. In addition, by the definition of the minimum and maximum degree, $\delta < d(v) < \Delta$ for every vertex v, so

   $$\sum_{v \in V(G)} \delta < \sum_{v \in V(G)} d(v) < \sum_{v \in V(G)} \Delta$$

   Since δ and Δ are constants and $|V(G)| = n$, we can show therefrom that

   $$n\delta < 2m < n\Delta$$

   from which the statement we were asked to prove follows easily.

5. *Either give an example of a 3-regular graph on 7 vertices, or prove it cannot exist.*

   Suppose such a graph G exists. We know that $\sum_{v \in V(G)} d(v) = 2\|G\|$. If a graph G is 3-regular, than each $d(v)$ is 3, so $\sum_{v \in V(G)} d(v) = \sum_{v \in V(G)} 3 = 3|V(G)| = 3 \cdot 7 = 21$. Thus $\|G\| = \frac{21}{2}$, a patent absurdity.