1. *Let G_K be a graph with 64 vertices, one for each square of an 8×8 chessboard, and let two vertices be adjacent if a chess king (capable of moving a single square orthogonally or diagonally) could move from one to the other. Let G_Q, G_R, G_B, and G_N be similarly constructed, but with adjecency determined by queen's moves (movement any distance orthogonally or diagonally), rook's moves (movement any distance orthogonally only), bishop's moves (movement any distance diagonally only), or knight's moves (movement 2 squares in one direction and one square perpendicularly).*

   (a) *Which of these graphs are connected? For those of these graphs that are not connected, how many connected components do they have?*

   Since the vertices associated with squares are adjacent if they are a single move away from eachother, a walk (which is a series of adjacent vertices) is associated with each series of moves on the chessboard. Thus, the existence of a walk between two vertices is identical to the existence of a sequence of moves between two squares; thus, graphs are connected if it is possible to move from any square to any other square in some number of moves. This is clearly true for the king, queen, and rook, and somewhat less obviously true for the knight, so G_K, G_Q, G_R, and G_N are connected. However, G_B is *not* connected, since there is no sequence of moves which will relocate a bishop on a white square to a black square or vice versa. It in fact consists of two components: the set of vertices associated with the black squares, and the set of vertices associated with the white squares.

   (b) *What is the distance between the two furthest apart vertices in a single component of G_K, G_Q, G_R, or G_B? (hint: what does distance in this graph measure with respect to the game itself?)*

   As discussed above, a walk represents a series of moves. The distance between two vertices is thus the fewest number of moves necessary to move between the associated squares. A king can only move one square in each direction at a time, so it would take 7 moves to get from one side to the opposite side. It is fairly easy to show that a king can in fact get anywhere in 7 moves, so the diameter of G_K is 7.

   A queen or rook can easily get from any square to any other square in 2 moves, simply by first moving to the right row, and then to the right column. In addition, a queen and a rook clearly can not go to every location in a single move: consider two points a knight's-mvoe form each other. Thus the diameter of both G_Q and G_R is 2.

   G_B within a single component looks quite similar to G_R (this can be observed by rotating a board 45 degrees), and in fact also has diameter 2.

   (c) *What is the distance between the two furthest apart vertices in G_N?*

   This one can only be determined, to the best of my knowledge, by brute force solution. It is fairly easy to show that there is a path between opposite corners of length 6 and that there is no path of length less than 6 between them; it is much more difficult to guarantee that no two other squares on the board are further apart.

2. *The* diameter *of a graph G, sometimes denoted $d(G)$, is the distance between the two furthest apart vertices in the graph, i.e.*

$$d(G) = \max_{u,v \in V(G)} d_G(u,v)$$

The radius *of a graph, denoted $r(G)$ is the minimum distance from a vertex v to the furthest vertex away from it, i.e.*

$$r(G) = \min_{v \in V(G)} \max_{u \in V(G)} d_G(u,v)$$

(the vertex v which has minimum distance from its furthest vertex is sometimes called a center *of the graph, keeping with the circle metaphor).*

(a) *Prove that $r(G) \leq d(G) \leq 2r(G)$.*

If we label $f(v) = \max_{u \in V(G)} d_G(u,v)$, then the radius of G is the minimum value of $f(v)$, and the diameter is the maximum value of $f(v)$. Trivially, it thus follows that $r(G) \leq d(G)$.

Let w be a center of the graph, so that $d(w,u) \leq r(G)$ for all vertices u. By the definition of the diameter, there must be a pair of vertices whose distance from each other is the diameter; let us call these vertices u and v. Then, using the triangle inequality and the observed property of the graph center w:

$$d(G) = d(u,v) \leq d(u,w) + d(v,w) \leq r(G) + r(G) = 2r(G)$$

(b) *Find a nontrivial graph in which $r(G) = d(G)$.*

Any graph in which all vertices are identical (called *vertex-transitive* graphs) will suffice. In particular, any cycle or complete graph will have identical radius and diameter. Specifically, $r(K_n) = d(K_n) = 1$, and $r(C_n) = d(C_n) = \lfloor \frac{n}{2} \rfloor$

(c) *Find a nontrivial graph in which $d(G) = 2r(G)$.*

Two simple examples of this property are paths on an odd number of vertices, or the "star graphs" defined by the family of complete bipartite graphs $K_{1,n}$. Specifically, $r(P_{2n+1}) = n$ while $d(P_{2n+1}) = 2n$, and $r(K_{1,n}) = 1$ while $d(K_{1,n}) = 2$.

3. **Bonus:** *Suppose that a graph G contains a cycle C, and that there is a path in G of length k between some two vertices of C . Show that G contains a cycle of length at least $\sqrt{k}$.*

Let P be the path escribed in the question. P shares at least two vertices with C: its start vertex and its final vertex. If P has vertices v_0, v_1, v_2, $\ldots$, v_k, let us denote the vertices where P intersects C with $v_0 = v_{i_1}$, v_{i_2}, v_{i_3}, $\ldots$, $v_{i_{\ell-1}}$, $v_{i_\ell} = v_k$, with $i_1 < i_2 < \cdots < i_\ell$. Two observations follow from this explicit identification of the intersection points of P with C:

First of all, C has length of at least ℓ, since $|C| \geq |C \cap P| = \ell$. Second, and more subtly, it is possible to find a cycle of length $\frac{k}{\ell-1} + \frac{\ell}{2}$. We demonstrate this by considering the path segments P_1 from $v_0 = v_{i_1}$ to v_{i_2}, P_2 from v_{i_2} to v_{i_3}, and so forth up to $P_{\ell-1}$

from $v_{i_{\ell-1}}$ to $v_{i_\ell} = v_k$. Since the concatenation of these paths has length k, the average length of one of these $\ell - 1$ paths is $\frac{k}{\ell-1}$. Some path must have at least average length, so one of these paths P_j has length of at least $\frac{k}{\ell-1}$. Since P_j only intersects C in the points v_{i_j} and $v_{i_{j+1}}$, one can form a cycle by taking P together with either of the two paths on C between v_{i_j} and $v_{i_{j+1}}$. Since C has length ℓ as noted above, at least one of these paths has length at least $\frac{\ell}{2}$. Thus, we can identify a cycle of length $\frac{k}{\ell-1} + \frac{\ell}{2}$ in G.

Since we have a cycle of length ℓ and a cycle of length $\frac{k}{\ell-1} + \frac{\ell}{2}$, we can guarantee a cycle of length at least $\max(\ell, \frac{k}{\ell-1} + \frac{\ell}{2})$. Some casewise algebra will suffice to show that this must be at least $\sqrt{k}$: either $\ell \geq \sqrt{k}$, in which case the expression above is clearly at least $\sqrt{k}$, or $\ell < \sqrt{k}$, in which case $\frac{k}{\ell-1} + \frac{\ell}{2} > \frac{k}{\sqrt{k}} = \sqrt{k}$.

A somewhat better bound than $\sqrt{k}$ is actually possible: $\max(\ell, \frac{k}{\ell-1} + \frac{\ell}{2})$ reaches its least value when $\ell = \frac{k}{\ell-1} + \frac{\ell}{2}$, which occurs when $\ell = \frac{1+\sqrt{8k+1}}{2} \approx \sqrt{2k}$