1. **(10 points)** *Let B_1 and B_2 be two blocks of a connected simple graph G. Prove that if B_1 and B_2 have a common vertex v, the graph $G - v$ produced by removing v from G is disconnected.*

   If two blocks overlap in a single vertex, each must consist of more than a single vertex (if exactly one of them were a single vertex, it would be strictly contained within the other and thus not maximal; if both of them were a single vertex, they would be identical and thus not be distinct). Thus, for intersection point u, $B_1 - v$ and $B_2 - v$ are nonempty and disjoint.

   Suppose $G - v$ was connected. Now consider neighbors of v called u_1 and u_2 in $B_1 - v$ and $B_2 - v$ respectively. By connectedness, there is a path P in $G - v$ between u_1 and u_2. Since blocks partition the edges (as seen in class), each edge of the path P lies in exactly one block; let the collection of blocks in which edges of P appear be denoted $\mathcal{B}$. We shall show that $\mathcal{B}$ must be 2-connected, which would violate maximality of B_1 and B_2.

   We know that in $\mathcal{B} - v$ there is a path (the aforementioned P) from u_1 to u_2, which, by the construction of $\mathcal{B}$, passes through every block of $\mathcal{B}$. Then $P + u_2vu_1$ is a cycle in $\mathcal{B}$. We shall now show, utilizing this cycle, that there are two disjoint paths between vertices v and w of $\mathcal{B}$. Suppose that v and w lie in two blocks C and D of $\mathcal{B}$. Since $P + u_2vu_1$ utilizes an edge from every block, there must be edges of C and D appearing in $P + u_2vu_1$. The cycle $P + u_2vu_1$ can be cut into two disjoint paths by removal of these edges. Together with disjoint paths within these blocks (guaranteed by 2-connectivity), these block-traversing paths will form a pair of disjoint paths in $\mathcal{B}$ between v and w.

2. **(20 points)** Given a connected simple graph G, let a graph H be produced in the following manner: let the blocks $B_1, B_2, \ldots, B_k$ of the graph G be associated with vertices $v_1, v_2, \ldots, v_k$ of H, and let v_i and v_j be adjacent if and only if B_i and B_j have a common vertex. Prove the following two facts:

   (a) **(10 points)** *H is connected.*

   We shall show that connectivity of H follows from connectivity of G (and vice versa, which is not necessary, but is fairly straightforward). Given that G is connected, consider blocks B_1 and B_2 of G, and vertices v_1 and v_2 respectively therefrom. By connectivity of G, there is a path in G from v_1 to v_2. Since the blocks partition the edges of G, each edge in the path lies in a single block, so the path traverses a sequence of (not necessarily distinct) blocks. Since any two consecutive blocks in this sequence must share the vertex utilized by two consecutive edges, the sequence of blocks above is in fact associated with a sequence of adjacent vertices in H. Since the first block in the sequence is B_1 and the last is B_2, there is a walk from B_1's associated vertex in H to B_2's. Since this is true for all blocks and associated vertices, it follows that H is connected.

   Conversely, if H is connected, then for vertices u_1 and u_2 in G, given that they lie in blocks B_1 and B_2 in connected H, there is a sequence of blocks associated with a path from B_1's associated vertex to B_2's; u_1 is connected to every vertex of B_1 by connectivity of a block; through shared vertices, it is also connected to every vertex of the other blocks associated with the path. Thus u_1 is connected to u_2 so G is connected.

(b) **(10 points)** *H is acyclic (the result from problem #1 may be useful here, and may be assumed to be true).*

If H contained any cycles, consider the collection of blocks $\mathcal{B}$ associated with the vertices of the cycle. Removal of the common vertex between two blocks associated with adjacent vertices would result in removal of the edge between the two blocks' associated vertices in H — which would leave the vertices associated with the blocks connected, as a cycle is 2-connected and 2-edge-connected. Thus, $\mathcal{B} - v$ would in this case still be connected, since as demonstrated in part (a) of this problem, connectivity in H is identical to connectivity in G. This would violate the known result of removing a shared vertex, shown to be true in problem #1.

3. **(10 points)** *We showed in class that the connectivity $\kappa(G)$ of a graph was no more than its minimum degree $\delta(G)$ — but in fact it can be far smaller! Demonstrate that a simple graph can have arbitrarily high minimum degree and still have very small connectivity.*

There are many ways to do this, but one of the most straightforward is to let G be a disjoint union of two K_ns: then G has $2n$ vertices, $\delta(G) = n - 1$, and since it is not connected at all, $\kappa(G) = 0$. There are other easily produced less extreme examples.

4. **(10 points)** *Let a connected (not necessarily simple) graph G have subgraphs G_1, G_2, ..., G_k such that each edge of G lies in exactly one G_i (that is, the graphs $G_1, \ldots, G_k$ partition the edges of G). Prove that if there is an Eulerian circuit on each of the G_i, there is an Eulerian circuit on G. Is the converse true?*

This is easiest demonstrated via a degree-measuring argument. Since every edge appears in exactly one G_i, the edges in G incident to a vertex can be counted by adding up edges incident to that vertex in every graph it appears in: thus, for any vertex $v \in V(G)$, we can assert that

$$d_G(v) = \sum_{V(G_i) \ni v} d_{G_i}(v)$$

Since each G_i contains an Eulerian tour, we know that every $d_{G_i}(v)$ above is even. Thus their sum is even, so each $d_G(v)$ is even, and thus G contains an Eulerian tour.

The converse is clearly not true, since an even sum need not be the result of a sum of even terms. In particular, a straightforward counterexample is $G = C_3$, with G_1, G_2, and G_3 consisting of a single edge each: G has an easy Eulerian tour, but none of the G_is do.

5. **(5 point bonus)** *The complement G^c of a graph G is a graph on the same vertex set such that vertices u and v are adjacent in G^c if and only if they are non-adjacent in G. Prove that either G or G^c is connected.*