1. **(10 points)** *Recall that G^c is the graph in which all non-adjacent vertices of G are adjacent and vice versa. Prove that if G has 11 or more vertices, G and G^c cannot both be planar.*

   For brevity, let us denote the number of vertices of G by n. Since the edge-sets of G and G^c are by definition disjoint and together account for every possible edge in among n vertices, $\|G\| + \|G^c\| = \frac{n(n-1)}{2}$. However, if G and G^c are planar, we know via the upper bound on number of edges in a planar graph that for $n \geq 3$, both $\|G\|$ and $\|G^c\|$ are no more than $3n - 6$. Thus, in order for G and G^c to be planar, it must be the case that $6n - 12 \geq \|G\| + \|G^c\| = \frac{n(n-1)}{2}$. This quadratic inequality can be simplified to $n^2 - 13n + 24 \leq 0$, which is true only if $n \leq \frac{13+\sqrt{73}}{2} \approx 10.7$. Thus both G and G^c cannot be planar if $n \geq 11$.

   Incidentally, a 1962 paper of Battle, Harary, and Kodama proves a stronger result: G and G^c cannot both be planar if G has 9 or more vertices. It is not too difficult, however, to construct 8-vertex planar graphs with planar complements.

2. **(5 points)** *Prove that for all graphs G, $\alpha(G) + \omega(G) \leq n + 1$.*

   Let A and W be the vertices respectively of a maximum-size independent set and a maximum-size clique in G, so by definition $|A| = \alpha(G)$ and $|W| = \omega(G)$. It is easy to show that $|A \cap W| \leq 1$, since if there were two vertices u and v in $A \cap W$, then, since u and v are both in a clique, they must be adjacent, and since they are in an independent set, they must be nonadjacent. Then, armed with the fact that these sets overlap in at most one point, we can observe that:

   $$n = |G| \geq |A \cup W| = |A| + |W| - |A \cap W| \geq \alpha(G) + \omega(G) - 1$$

   so $\alpha(G) + \omega(G) \leq n + 1$.

3. **(35 points)** *The following is a somewhat simpler construction than that shown in class for constructing graphs with low clique number and high coloring number: for a graph G with vertices $v_1, \ldots, v_n$, let $M(G)$ be a graph with vertex-set $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ (for $2n + 1$ vertices total). For every edge $v_i v_j$ in G, include the following edges in $M(G)$: $x_i x_j$, $x_i y_j$, $x_j y_i$. In addition, include the edges $y_1 z$, $y_2 z$, $y_3 z$, $\ldots$, $y_n z$.*

   This construction is known as the *Mycielskian*, and was developed by Jan Mycielski in 1955.

   (a) **(5 points)** *Show that for any graph G, $\chi(M(G)) \leq \chi(G) + 1$.*

   Showing an upper bound on the coloring number is simply a matter of showing that a coloring in the given number of colors is possible (even if not optimal). Let $k = \chi(G)$; by definition G is k-colorable, so suppose we have a coloring function $c(v_i)$ mapping each vertex of G to one of the colors $1, \ldots, k$. We shall consider coloring $M(G)$ with $k + 1$ colors as such: let x_i and y_i have the colors $c(v_i)$, and give z the heretofore unused color $k + 1$.

   We shall show that this coloring of $M(G)$ is in fact one in which adjacent vertices are different colors: x_i is adjacent to x_j only if v_i is adjacent to v_j, so since G

is properly colored, $c(v_i) \neq c(v_j)$, and thus, since these two colors are also those used on x_i and x_j, x_i and x_j must be different colors. Likewise, if x_i is adjacent to y_j, v_i must be adjacent to v_j and the argument proceeds as above. All other possible adjacencies can be dispensed with easily: y_i adjacent to y_j is impossible by construction of $M(G)$, and z's adjacenct vertices must necessarily be of a different color than z, since z's chosen color is unique in the graph.

(b) **(10 points)** *Show that* $\chi(M(G)) \geq \chi(G) + 1$.

As above, let $k = \chi(G)$. An upper bound on coloring results from showing that the given bound is the fewest number of colors with which a coloring is possible: thus, the most straightforward way to prove this bound is to demonstrate that it is *not* possible to color $M(G)$ with only k colors. Suppose there is such a coloring c mapping the vertices of $M(G)$ to the colors $1, \ldots k$.

Without loss of generality we may assume $c(z) = k$, since any permutation of a coloring would have no effect on its validity. We know that, since the induced subgraph of $M(G)$ on the x_i vertices is isomorphic to G, we would need all k colors to color the x_i vertices, even if we only take into account adjacencies to other x_j. We shall show that the coloring scheme described here, even as vaguely-described as was done here, can not actually be a valid vertex-coloring.

Consider the set of vertices x_i such that $c(x_i) = k$. Since we demonstrated above that k colors are necessary to color all the x_i, we know this set must be nonempty; furthermore, since there is no way to recolor these vertices to eliminate the color k outright, at least one of these vertices must be *forced* to be of color k, so it must be adjacent to vertices x_j in each of the colors $1, 2, \ldots, k-1$. Thus the associated vertex y_i must, by construction of $M(G)$ be adjacent to the same vertices x_j, forcing it to also be of color k. Then y_i and its neighbor z are both of color k, which is disallowed.

(c) **(10 points)** *Show that if G does not have a K_3 subgraph, neither does $M(G)$.*

This can straightforwardly be done by an exhaustive consideration of the various sets from which 3 points can be drawn, and demonstration that in no case are all 3 vertices adjacent.

Consider vertices x_i, x_j, and x_k for distinct i, j, and k. These are adjacent if and only if v_i, v_j, and v_k are adjacent, which would require that G have a K_3 subgraph, which is not possible.

Consider vertices x_i, x_j, and y_k, in which i and j are distinct. If $k = i$ or $k = j$, then y_k is not adjacent to x_k, and these three vertices cannot thus be mutually adjacent. If, on the other hand, k is distinct from both i and j, then the edges $x_i y_k$, $x_j y_k$, and $x_i x_j$ exist in $M(G)$ if and only if $v_i v_k$, $v_j v_k$, and $v_i v_k$ exist in G, which would depend on the existence of a K_3 subgraph in G, which was given not to exist.

Consider distinct vertices y_i, y_j, and any third vertex of $M(G)$; since y_i is not adjacent to y_j, these vertices do not form a K_3 subgraph.

Finally, consider distinct vertices z, x_i, and any third vertex of $M(G)$; since x_i is not adjacent to z, these vertices do not form a K_3 subgraph.

These four cases can be easily seen to account for every possible selection of three vertices from $M(G)$.

(d) **(5 points)** Let $M_2 = K_2$ and $M_i = M(M_{i-1})$ for $i > 2$. Prove using the results from the above sections that $\chi(M_i) = i$ and $\omega(M_i) = 2$.

We shall prove by induction that $\chi(M_i) = i$ and $\omega(M_i) = 2$.

Our base case is $M_2 = K_2$, which can easily be demonstrated to have chromatic number 2 and clique number 2.

For our inductive step, we may take it as given that $\chi(M_{i-1}) = i - 1$ and that $\omega(M_{i-1}) = 2$. By parts (a) and (b) above we have that $\chi(M(G)) = \chi(G) + 1$, so $\chi(M_i) = \chi(M(M_{i-1})) = \chi(M_{i-1}) + 1 = (i - 1) + 1 = i$.

Similarly, using our inductive hypothesis that $\omega(M_{i-1}) = 2$, we may assert that M_{i-1} has no K_3 subgraph (since M_{i-1}'s largest clique has size 2). Thus, by part (c) above, $M(M_{i-1}) = M_i$ also has no K_3 subgraph, so $\omega(M_i) = 2$.

(e) **(5 points)** *Prove that M_i has $3 \cdot 2^{i-2} - 1$ vertices.*

By construction we know that $|M(G)| = 2|G| + 1$. This allows for a natural proof of the above assertion by induction.

For our base case $i = 2$, we can see that $|M_2| = |K_2| = 2$, which is indeed equal to $3 \cdot 2^0 - 1$.

For the inductive step, let us assume that $|M_{i-1}| = 3 \cdot 2^{i-3} - 1$. Then, we find that

$$|M_i| = |M(M_{i-1})| = 2|M_{i-1}| + 1 = 2\left(3 \cdot 2^{i-3} - 1\right) + 1 = 3 \cdot 2^{i-2} - 2 + 1 = 2^{i-2} - 1$$

4. **(5 point bonus)** *If G is a planar graph with n vertices without any C_3 subgraphs, what is the maximum number of edges that G can have? What if G lacks both C_3 and C_4 subgraphs?*

If G lacks C_3 subgraphs, then every face must be bounded by at least 4 edges. Since every edge bounds at most 2 faces, we thus have that $4f \leq 2\|G\|$, or that $f \leq \frac{1}{2}\|G\|$. Using Euler's formula, we know that $n + f - \|G\| = 2$, so that $n + \frac{1}{2}\|G\| - \|G\| \geq 2$. This simplifies algebraically to $\|G\| \leq 2n - 4$.

Likewise, if G lacks C_3 and C_4 subgraphs, then every face must be bounded by at least 5 edges. Since every edge bounds at most 2 faces, we thus have that $5f \leq 2\|G\|$, or that $f \leq \frac{2}{5}\|G\|$. Using Euler's formula, we know that $n + f - \|G\| = 2$, so that $n + \frac{2}{5}\|G\| - \|G\| \geq 2$. This simplifies algebraically to $\|G\| \leq \frac{5}{3}(n - 2)$.