Due on Wednesday, April 1.

1. **(15 points)** *Let* $f(r) = R(\underbrace{3, 3, \ldots, 3}_{r \text{ terms}})$*; that is, let* $f(r)$ *be the least value of* n *such that coloring a* K_n *with* r *colors guarantees a monochromatic* K_3*. For example, using results seen in class, we know that* $f(2) = R(3, 3) = 6$ *and* $f(3) = R(3, 3, 3) = 17$*.*

   (a) **(10 points)** *Prove that* $f(r) \leq r(f(r-1) - 1) + 2$*.*

   For brevity, let $n = r(f(r-1) - 1) + 2$. Then, let us consider a coloring of the edges of K_n with r different colors. Given a specific vertex v of K_n, we know that its degree is $n-1$, so it will have $r(f(r-1)-1)+1$ edges adjacent to it, each of them in one of r colors. The average number of these edges per color is thus $\frac{r(f(r-1)-1)+1}{r} = f(r-1) - 1 + \frac{1}{r}$, so since some color must occur at least the average number of times, there is some color which is used at least $f(r-1)$ times among the edges incident to v (the Pigeonhole Principle could also be used to justify this assertion).

   Without loss of generality, let us call this color "red" (the name is inconsequential, but assigning a name to it makes the rest of this argument more readable). Thus we know that there are vertices $u_1, u_2, \ldots, u_{f(r-1)}$ such that the edge vu_i is red for each u_i. Now let us consider the ways in which the edges of the $K_{f(r-1)}$ induced by the vertices u_i are colored. If any edge u_iu_j is red, then we have a monochromatic K_3 in red among the vertices v, u_i, and u_j. If, on the other hand, none of the edges of this $K_{f(r-1)}$ are red, then this $K_{f(r-1)}$ is colored in $r-1$ colors (the r colors we had at our disposal, with red removed), and by definition of f, such a coloration guarantees a monochromatic K_3. Thus, regardless of how we color a K_n with r colors, we are guaranteed a monochromatic K_3, and thus $f(r) \leq n$.

   This bound, incidentally, is known to be imperfect, but exactly how imperfect is still unknown. The above result gives $f(4) \leq 66$; the best bounds are not much better, giving $51 \leq f(4) \leq 62$.

   (b) **(5 points)** *Use the above result to give* $f(r)$ *the explicit bound* $f(r) \leq 3r!$ *(where* $n! = n(n-1)(n-2)\cdots(3)(2)(1)$*.*

   We can prove this by induction quite easily. For the base case, note that $f(2) = 6 = 3 \cdot 2!$. For the inductive step, let us assume that $f(r-1) \leq 3(r-1)!$. Then, using the result from above,

   $$f(r) \leq r(f(r-1)-1)+2 = rf(r-1) - r + 2 \leq r(3(r-1)!) + (2-r) < r(3(r-1)!) = 3r!$$

2. **(15 points)** *The Ramsey number* $R(K_k, K_{1,\ell})$ *is the least* n *such that a coloring of* K_n *in red and blue has neither a red* K_k *subgraph nor a blue* $K_{1,\ell}$ *subgraph (the complete graphs* $K_{1,\ell}$ *are sometimes called the* star graphs*, as they consist of a single hub vertex and many others joined to it).*

   (a) **(5 points)** *Describe a construction to prove that* $R(K_k, K_{1,\ell}) > (k-1)\ell$*.*

   This construction is a special case of one explored in class. Let us color the edges of a $K_{(k-1)\ell}$ in the following manner: divide the vertices into classes $S_1, S_2, S_3, \ldots, S_{k-1}$, with each class consisting of ℓ vertices. Color all edges between vertices in the same class in blue, and all edges between vertices in different classes in red. This construction, we assert, contains neither a red K_k nor a blue $K_{1,\ell}$.

It cannot contain a blue $K_{1,\ell}$ since, in order to do so, some vertex must be have ℓ incident blue edges. We know that each vertex is adjacent by blue edges only to other members of its class; since each class contains ℓ vertices, each vertex is adjacent via blue edges to exactly $\ell - 1$ other vertices.

Likewise, it cannot contain a red K_k, since if we select k vertices belonging to $k - 1$ classes, two of them must be in the same class and thus be connected by a blue edge, not a red edge, so this arbitrarily selected set of k vertices cannot be the vertices of a red K_k.

(b) **(10 points)** *Prove that $R(K_k, K_{1,\ell}) \leq (k-1)\ell+1$; that is, show that any coloring of the edges among $(k-1)\ell+1$ points necessarily has a red K_k subgraph or a blue $K_{1,\ell}$ subgraph.*

This question is quite difficult and will require an inductive argument on k.

It is easy to show this result for the base case $k = 2$, since avoiding a red K_2 would only be possible if every edge were blue, and $K_{1,\ell}$ is a subgraph of $K_{\ell+1}$, so $R(K_2, K_{1,\ell}) = \ell + 1 = (2 - 1)\ell + 1$.

In the inductive step, we may assume that $R(K_{k-1}, K_{1,\ell}) \leq (k - 2)\ell + 1$. Let $n = (k - 1)\ell + 1$ for brevity. We shall show that every coloring of K_n contains either a red K_k or a blue $K_{1,\ell}$, and thus that $R(K_k, K_{1,\ell}) \leq n$. Consider a specific vertex v of K_n. v has $n - 1$ incident edges, each of which must be red or blue. If ℓ or more edges incident to v are blue, then we immediately have a blue $K_{1,\ell}$ with v as its hub.

On the other hand, if fewer than ℓ of the $n - 1$ edges incident on v are blue, than more than $(n - 1) - \ell$ edges incident on v are red. Thus at least $n - \ell = (k - 1)\ell + 1 - \ell = (k - 2)\ell + 1$ of the edges incident on v are red. The vertices $u_1, u_2, \ldots, u_{(k-2)\ell+1}$ which are adjacent to v via these edges induce a $K_{(k-2)\ell+1}$ whose edges are all colored blue or red. By the inductive hypothesis, in the coloring of this $K_{(k-2)\ell+1}$, either a red K_{k-1} or a blue $K_{1,\ell}$ appears. In the former case, this red K_{k-1}, together with the red edges from all of its vertices to v, forms a red K_k in the larger graph; in the latter case, we definitively have a blue $K_{1,\ell}$. Thus, every coloring of K_n contains either a red K_k or a blue $K_{1,\ell}$.

The results above are a special case of Proposition 9.2.1 in Diestel; $K_{1,\ell}$ can be replaced by an arbitrary tree T, and ℓ replaced by $|T| - 1$.

3. **(10 points)** *Prove that* $\operatorname{ex}(n, K_{1,\ell}) = \lfloor \frac{n(\ell-1)}{2} \rfloor$.

It can be easily seen that G has $K_{1,\ell}$ as a subgraph if and only if some vertex in G has degree ℓ or more. Thus, in order to avoid a $K_{1,\ell}$, every vertex must have degree of $\ell - 1$ or less. Since $2\|G\| = \sum_{v \in V(G)} d(v)$, we thus know that if G is $K_{1,\ell}$-free, then $\|G\| \leq \frac{|G|(\ell-1)}{2}$; Since the number of edges in a graph must be an integer, it thus follows that $\operatorname{ex}(n, K_{1,\ell}) \leq \lfloor \frac{n(\ell-1)}{2} \rfloor$. To show this is an equality, we must simply demonstrate the existence of a graph with n vertices, $\lfloor \frac{n(\ell-1)}{2} \rfloor$ edges, and maximum degree less than ℓ. The Harary graph $H_{n,\ell-1}$, which we saw when exploring connectivity, is such an example, althoguh there are others which will suffice.

4. **(10 points)** *Using Turán's theorem, prove that* $\lim_{n\to\infty} \frac{\mathrm{ex}(n,K_r)}{\|K_n\|} = \frac{r-2}{r-1}$*. Note: you will need to justify your approximate counts for the number of edges between various parts of* $T_{r-1}(n)$*; you may assume* n *divides evenly by* $r-1$ *to simplify the arithmetic.*

Assuming n divides evenly by $r-1$, $T_{r-1}(n)$ is an $(r-1)$-partite graph in which each part contains exactly $\frac{n}{r-1}$ vertices. The number of edges is the nubmer of ways to select two vertices from different parts. We may enumerate these by multiplying the number of ways to select the following three parameters: two distinct parts, which can be done in $\frac{(r-1)(r-2)}{2}$ ways; avertex from the first part, which can be done in any of $\frac{n}{r-1}$ ways, and a vertex from the second part, which can also be done in any of $\frac{n}{r-1}$ ways. Thus, there are $\frac{(r-1)(r-2)}{2}\frac{n}{r-1}\frac{n}{r-1}$ edges in $T_{r-1}(n)$, and thus by Turán's theorem $\mathrm{ex}(n,K_r) = t_{r-1}(n) = \frac{n^2(r-2)}{2(r-1)}$. Then:

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,K_r)}{\|K_n\|} = \lim_{n\to\infty} \frac{\frac{n^2(r-2)}{2(r-1)}}{\frac{n(n-1)}{2}} = \lim_{n\to\infty} \frac{(r-2)n^2}{(r-1)(n^2-n)} = \frac{r-2}{r-1}$$

5. **(5 point bonus)** *Explore* $\mathrm{ex}(n,P_4)$ *— that is, the greatest number of edges that can exist among* n *vertices without producing a path of length 3. What bounds can you place on this quantity?*

Experimentation is sufficient to discover several values of $\mathrm{ex}(n,P_4)$:

n	1	2	3	4	5	6
$\mathrm{ex}(n,P_4)$	0	1	3	3	4	6

There are several easy observations: for instance, $\mathrm{ex}(n,P_4) \geq n-1$, since the star graph $K_{1,n-1}$ has no path traversing more than 3 vertices; if n is divisible by 3, we can improve this to $\mathrm{ex}(n,P_4) \geq n$ by considering a graph consisting of several disconnected triangles.

In fact, it can be shown that the only connected P_4-free graphs are the isolated vertex, the star graphs $K_{1,k}$, and the triangle K_3. By a decomposition of an arbitrary graph G into connected components, it can be shown that the bounds determined above are the best possible.