Due on Wednesday, April 1.

1. **(15 points)** Let $f(r) = R(\underbrace{3, 3, \ldots, 3}_{r \text{ terms}})$; that is, let $f(r)$ be the least value of n such that coloring a K_n with r colors guarantees a monochromatic K_3. For example, using results seen in class, we know that $f(2) = R(3, 3) = 6$ and $f(3) = R(3, 3, 3) = 17$.

   (a) **(10 points)** Prove that $f(r) \leq r(f(r-1)-1)+2$.

   (b) **(5 points)** Use the above result to give $f(r)$ the explicit bound $f(r) \leq 3r!$ (where $n! = n(n-1)(n-2)\cdots(3)(2)(1)$.

2. **(15 points)** The Ramsey number $R(K_k, K_{1,\ell})$ is the least n such that a coloring of K_n in red and blue must have either a red K_k subgraph or a blue $K_{1,\ell}$ subgraph (the complete graphs $K_{1,\ell}$ are sometimes called the *star graphs*, as they consist of a single hub vertex and many others joined to it).

   (a) **(5 points)** Describe a construction to prove that $R(K_k, K_{1,\ell}) > (k-1)\ell$.

   (b) **(10 points)** Prove that $R(K_k, K_{1,\ell}) \leq (k-1)\ell+1$; that is, show that any coloring of the edges among $(k-1)\ell+1$ points necessarily has a red K_k subgraph or a blue $K_{1,\ell}$ subgraph.

3. **(10 points)** Prove that $\text{ex}(n, K_{1,\ell}) = \lfloor \frac{n(\ell-1)}{2} \rfloor$.

4. **(10 points)** Using Turán's theorem, prove that $\lim_{n\to\infty} \frac{\text{ex}(n,K_r)}{\|K_n\|} = \frac{r-2}{r-1}$. Note: you will need to justify your approximate counts for the number of edges between various parts of $T_{r-1}(n)$; you may assume n divides evenly by $r-1$ to simplify the arithmetic.

5. **(5 point bonus)** Explore $\text{ex}(n, P_4)$ — that is, the greatest number of edges that can exist among n vertices without producing a path of length 4. What bounds can you place on this quantity?