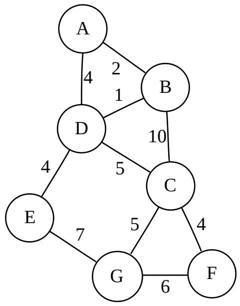
1. **(10 points)** *Using Dijkstra's algorithm, find a shortest path on this graph between A and G, showing your work:*

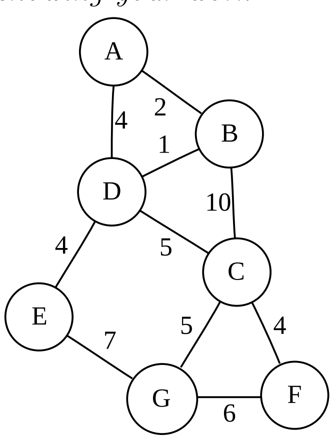


The following table shows the progression of Dijkstra's algorithm; subscripts, which you need not include, show the order in which things are filled in to the table.

Vertex	Visited?	Distance	Notes
A	✓$_1$	0_0	—$_0$
B	✓$_2$	2_1	A_1
C	 ✓$_5$	~~12_2~~ 8_3	~~AB_2~~ ABD_3
D	 ✓$_3$	~~4_1~~ 3_2	~~A_1~~ AB_2
E	✓$_4$	7_3	ABD_3
F	✓$_6$	12_5	$ABDC$
G	✓$_7$	~~14_4~~ 13_5	~~$ABDE_4$~~ $ABDC$

So in the final step we determine that the shortest path from A to G is of length 13, following the edges from A to B to D to C to G.

2. **(10 points)** *Using Kruskal's algorithm, find a minimum spanning tree on this graph, showing your work:*



We list the edges in order of weight, and include each in our spanning tree in turn if the vertices it is between are not already connected:

Edge	Weight	Useful?
BD	1	✓
AB	2	✓
AD	4	
DE	4	✓
CF	4	✓
CD	5	✓
CG	5	✓
FG	6	
EG	7	
BC	10	

The spanning tree consists of the edges chosen in the third column above. Your list tree may look slightly different if you listed edges of equal weight in a different order.

3. **(40 points)** *For each of the following statements, either prove it (if true) or give a counterexample (if false).*

   (a) *For every even n, a 3-regular simple graph on n vertices exists.*

   This one (to my surprise, as I didn't intend it) turns out not to be true! For $n = 2$, it is easy to demonstrate no 3-regular graph on n vertices exists. For $n > 2$, on the other than, these graphs exist (and are, for example, the Harary graphs $H_{n,3}$ we saw when discussing minimal 3-connected graphs.

   (b) *For a graph G and a specific vertex w thereof, if there is a path from w to every vertex $u \in V(G)$, then G is connected.*

   This can be shown to be true: the criterion for connectedness is that there is a path from every vertex u to every vertex v. Finding such a path, given the knowledge above, is a bit difficult, but finding a *walk* from u to v is easy: the condition given above guarantees existence of a path from u to w, and of a path from w to v, and if we join these paths end-to-end we get a walk (note: *not* necessarily a path!) from u to v. The existence of a walk between two points, as shown in class, guarantees a path between the same two points, so there is a path from u to v. Thus, since two arbitrary vertices of G are connected, G is connected.

   (c) *If G is a simple graph with n vertices and $n-1$ or more edges, then G is connected.*

   This is demonstrably false; the smallest graph violating it would be the disjoint union of a K_3 with an isolated point. This has 4 vertices, 3 edges, and is not connected.

   (d) *For any vertices u, v, and w in a tree, $d(u, v) + d(v, w) = d(u, w)$.*

   This is false: the simplest counterexample is P_3 with w between u and v, so $d(u, v) = 2$ and $d(v, w) = d(u, w) = 1$, so $d(u, v) + d(v, w) = 3$ and $d(u, w) = 1$.

4. **(5 point bonus)** *If a simple graph G has n vertices and n or more edges, then G contains a cycle.*

It is easy to show that a *connected* graph G with as many edges as vertices contains a cycle: if G is an acyclic connected graph, then it is a tree, and has one fewer edges than vertices; thus, a connected graph with the same number of vertices as edges is not acyclic and must therefore contain a cycle.

To prove that this is also true for disconnected graphs, consider G's decomposition into connected components C_1, C_2, ..., C_k. Since every edge and vertex lies in exactly one component, it is clear that

$$|C_1| + |C_2| + \cdots + |C_k| = |G| = n$$

and likewise

$$\|C_1\| + \|C_2\| + \cdots + \|C_k\| = \|G\| = n$$

Now, if it were the case for every i that $\|C_i\| < |C_i|$, then their sums would likewise be subject to this inequality, yielding $\|G\| < |G|$, which is not true. Thus, there must be at least one i such that $\|C_i\| \geq |C_i|$. This graph C_i is connected and has at least as many edges as vertices: our argument above for connected G can thus be applied to this graph to prove that it contains a cycle (and thus, so does the larger disconnected graph G).