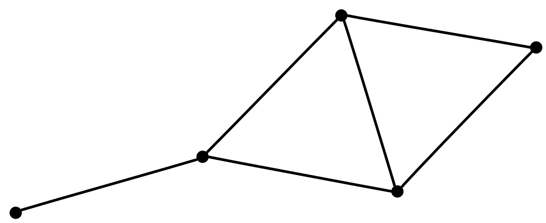
1. **(10 points)** *Find three distinct acyclic orientations of the following graph.*



There are several such based on different possible choices of vertex-orderings.

2. **(10 points)** *Describe (either in words or a diagram) a K_5-free graph on 7 vertices with as many edges as possible. How many edges does it have?*

This is a complete 5-partite (quintapartite) graph in which four parts have 2 vertices each and the fifth part has 3 vertices.

Between each of the pairs of 2-vertex parts there are $2 \cdot 2 = 4$ edges; there are 6 pairs of 2-vertex parts so there are $6 \cdot 4 = 24$ edges of this sort. In addition, there are edges from each of the vertices in the 3-vertex part to every vertex in the other parts, for $3 \cdot 8$ edges incident on the 3-vertex part. This graph thus has $24 + 24 = 48$ edges total.

3. **(40 points)** *For each of the following statements, either prove it (if true) or give a counterexample (if false).*

(a) *For any graph H, $\text{ex}(n, H) \leq \frac{1}{2}n^2$.*

True. $\text{ex}(n, H)$ measures the maximum number of edges in an H-free simple graph on n vertices; an H-free simple graph is, first and foremost, a simple graph, so it has no more edges than K_n. Thus, regardless of what H is, $\text{ex}(n, H) \leq \|K_n\| = \frac{n(n-1)}{2} < \frac{n^2}{2}$.

(b) *Every digraph D contains a vertex v such that $d^+(v) = d^-(v)$.*

False. One simple counterexample is an orientation of P_2, which contains two vertices, each of which has either an incoming or an outgoing edge. Many other counterexamples are possible.

(c) *If G has 5 vertices, then either $\omega(G) > 2$ or $\alpha(G) > 2$.*

False. $\omega(C_5) = 2$ and $\alpha(C_5) = 2$. Note that if G had six vertices, this statement would be true as a result of $R(3, 3) = 6$.

(d) *If graph G has n vertices, then G has $n!$ or fewer acyclic orientations. ($n!$ is defined as $n(n-1)(n-2)(n-3)\ldots(3)(2)(1)$; for example $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$).*

True. Every acyclic orientation can be produced (albeit not necessarily uniquely) by a labeling of the n vertices with numbers $1, \ldots, n$ and orienting the edges based on that ranking of vertices. Since there are only $n!$ ways to assign the numbers $1, \ldots, n$ to the vertices, there are $n!$ or fewer acyclic orientations.

4. **(5 point bonus)** *Prove that if all vertices in a tournament have different in-degrees, then the tournament is acyclic.*

We can prove this by induction (which is probably the easiest way to do so). What we will specifically prove by induction is the following: if T is a tournament on n vertices where every vertex has distinct indegrees, T has the specific acyclic orientation given by ranking the vertices in order by indegrees, so that if $d^+(u) > d^+(v)$, there is an edge oriented from v to u. This is trivially true on the 2-vertex tournament, to establish the base case.

To handle the inductive case, let us assume this property holds for a tournament on $n-1$ vertices. The minimum indegree of a vertex in an n-vertex tornament is 0, and the maximum indegree of a vertex is $n-1$ (if there are incoming edges from every single other vertex in the graph). Since there are only n integers between 0 and $n-1$ inclusive, the indegrees appearing in an n-vertex tournament in which all vertices have distinct indegree must be $0, 1, 2, \ldots, n-1$. Thus, there is a vertex v with indegree 0, so the edges between v and every other vertex of the graph are oriented away from v. Let $T' = T - v$. Since every vertex of T' has an edge incoming from v, $d^+_{T'}(u) = d^+_T(u) - 1$ for all $u \in T'$. Thus, the indegrees in T' are all distinct (since their degrees in T were distinct, and they will remain distinct upon being all decremented by 1). By induction, the edge-orientation is induced by indegrees in T', and thus will also be induced by indegrees in T, since the comparative values of two indegrees is unchanged upon incrementing both by 1. And finally, v can be included in this degree-induced ranking, since $d^+(v) < d^+(u)$ for all vertices $u \neq v$, and there is an edge from v to every $u \neq v$.