Please work on this independently. You are welcome to use any written resources (i.e. books, e-texts, etc.), but do not copy information verbatim or consult with people either in or out of this class.

1. **(50 points)** Draw a graph satisfying the conditions described, or briefly explain why such a graph does not exist.

   (a) **(10 points)** Simple graph G has 8 vertices, 12 edges, and is 3-connected.

   (b) **(10 points)** Tournament T has 5 vertices and is acyclic.

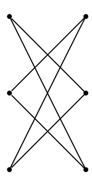
   (c) **(10 points)** Simple K_3-free graph G has 9 vertices and 15 edges.

   (d) **(10 points)** Simple graph G has 8 vertices, independence number 3 (i.e. $\alpha(G) = 3$) and chromatic number 2 (i.e. $\chi(G) = 2$).

   (e) **(10 points)** Directed graph D has 6 vertices, $d^+(v) = d^-(v) = 2$ at every vertex v, and has a directed Hamiltonian cycle.

2. **(50 points)** Explore the following properties of the below-defined operation.

   We define the *cross product* of simple graphs G_1 and G_2, denoted $G_1 \times G_2$, as a graph whose vertices are ordered pairs of vertices (v_1, v_2) from G_1 and G_2 respectively. (v_1, v_2) is adjacent to (u_1, u_2) in $G_1 \times G_2$ if and only if u_1 is adjacent to v_1 and u_2 is adjacent to v_2. For instance, below is the 6-vertex graph produced by taking the cross product $K_2 \times K_3$.



   (a) **(10 points)** Prove that if (u, v) is a vertex of $G \times H$, then its degree is $d_G(u)d_G(v)$.

   (b) **(10 points)** Prove that the graph $G \times H$ is $\chi(G)$ colorable, and thus that $\chi(G \times H) \leq \min(\chi(G), \chi(H))$.

   (c) **(10 points)** Demonstrate that even if G and H are both connected, $G \times H$ need not be.

   (d) **(10 points)** Show that $G \times H$ has an independent set of size $\alpha(G)\alpha(H)$, and thus that $\alpha(G \times H) \geq \alpha(G)\alpha(H)$.

   (e) **(10 points)** Show that if $G \times H$ is connected, and at least one of G or H has an Eulerian tour, then $G \times H$ has an Eulerian tour.

   (f) **(6 point bonus)** Part (b) put an upper bound on $\chi(G \times H)$. Can you improve this bound, or develop a lower bound?