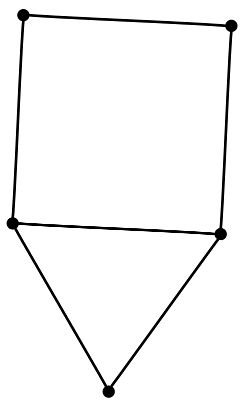
1. **(10 points)** *Describe (either in words or a diagram) a K_6-free graph on 11 vertices with as many edges as possible. How many edges does it have?*

   This is a complete 5-partite (quintapartite) graph in which four parts have 2 vertices each and the fifth part has 3 vertices.

   Between each of the pairs of 2-vertex parts there are $2 \cdot 2 = 4$ edges; there are 6 pairs of 2-vertex parts so there are $6 \cdot 4 = 24$ edges of this sort. In addition, there are edges from each of the vertices in the 3-vertex part to every vertex in the other parts, for $3 \cdot 8$ edges incident on the 3-vertex part. This graph thus has $24 + 24 = 48$ edges total.

2. **(10 points)** *Find three distinct acyclic orientations of the following graph.*



3. There are several such based on different possible choices of vertex-orderings.

4. **(40 points)** *For each of the following statements, either prove it (if true) or give a counterexample (if false).*

   (a) *There is no digraph D such that $d^+(u) \leq d^-(u)$ for all vertices of D.*

   This is false as written (because I was careless and wrote $\leq$ instead of $<$); One can produce several graphs in which $d^+(u) = d^-(u)$ everywhere.

   The intended statement with $<$ is true, since if the inequality were true on all vertices, it would be true on their sums, so $\sum_{u \in V(D)} d^+(u) < \sum_{u \in V(D)} d^-(u)$, which would be at odds with the necessary fact about digraphs that $\sum_{u \in V(D)} d^+(u) = \sum_{u \in V(D)} d^-(u)$

   (b) *There is a value of n such that the random graph $G(n, \frac{1}{2})$ has a probability of at least 75% of containing C_6 as an induced subgraph.*

   We know that for any 6 vertices of $G(n, \frac{1}{2})$, the probability that they form an induced C_6 subgraph with the vertices in a particular order is $\left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^{(15-6)} = \left(\frac{1}{2}\right)^{15}$. Thus, the probability that these particular vertices do not contain an induced C_6 is no more than $1 - \left(\frac{1}{2}\right)^{15}$. Taking $\frac{n}{6}$ distinct 6-vertex sets,w e see that the porbability that none of them contain an induced C_6 is less than $\left(1 - \left(\frac{1}{2}\right)^{15}\right)^{n/6}$. Since $1 - \left(\frac{1}{2}\right)^{15} < 1$, this can be made as small as desired by choice of a sufficiently large n; specifically, it can be made to be less than 25%.

(c) *If D is an acyclic orientation of a graph G and G has a Hamiltonian cycle, then D has a directed Hamiltonian path.*

This is false: consider the orientation of C_4 in which two opposite corners have indegree two, and the other two corners have outdegree 2. C_4 has a Hamiltonian cycle, but this orientation has no path of length more than 1.

(d) *If $||G|| \geq \frac{|G|}{2}$, then G has P_3 as a subgraph.*

Once again my absurdity with inequalities foils me! This is false as written, since $\frac{n}{2}$ disconnected copies of P_2 are in fact a P_3-free graph with $\frac{n}{2}$ edges.

If, on the other hand, $\geq$ were replaced with $>$, this would be true. We know, from way back when, that $\Delta(G) \geq \frac{2\|G\|}{|G|} > \frac{|G|}{|G|} = 1$, so $\Delta(G) \geq 2$. Thus some vertex u in G has degree 2 or more, so u has neighbors v and w; vuw is a path of length 2, and thus a P_3 subgraph.