1. **(20 points)** *Answer the following questions concerning applications of algebra.*

   (a) **(8 points)** *At noon, András starts walking westwards from campus at 4 miles per hour; at 2 PM, Borbála starts bicycling east from campus at 12 miles per hour. If t is the number of hours that have passed since noon, give an expression for the distance between András and Borbála in terms of t.*

   When t hours have passed since noon, András has walked westwards for t hours at 4 miles per hour, and will be $4t$ miles to the west of campus; at the same time, Borbála will have been bicycling for only $t-2$ hours, but at a rate of 12 miles per hour, so she will be $12(t-2)$ miles to the east of campus. The distance between them is the sum of their distances, since they traveled in opposite directions: thus this expression will be $4t + 12(t-2)$, or, if you prefer, $16t - 24$.

   (b) **(12 points)** *If you have access to unlimited quantities of two different brine solutions which are 30% salt and 80% salt, how much of each solution should you mix together to get 10 liters of a 40%-salt brine?*

   Let x denote the number of liters of the 30% solution used. Then $10 - x$ liters of the 80% solution are used. The 30% solution will contribute 0.3 liters of salt for each liter of solution, so it contributes $0.3x$ liters of salt in total; likewise, the 80% solution contributes $0.8(10 - x)$ liters of salt. We want a solution which contains $10 \cdot 40\% = 4$ liters of salt, so the combination of the mixtures should satisfy

$$\begin{aligned} 0.3x + 0.8(10 - x) &= 4 \\ 0.3x + 8 - 0.8x &= 4 \\ -0.5x &= -4 \\ x &= 8 \end{aligned}$$

   so the mixture should use 8 liters of the 30% brine, and 2 liters of the 80%.

2. **(15 points)** *Simplify the following expressions.*

   (a) **(5 points)** $(3 + 4i)(1 - 2i) + (2 - i)$.

   We perform a fairly standard expansion of the product and sum, and make use of the fact that $i^2 = -1$:

$$\begin{aligned} (3 + 4i)(1 - 2i) + (2 - i) &= 3 - 6i + 4i - 8i^2 + 2 - i \\ &= 5 - 3i - 8(-1) = 13 - 3i \end{aligned}$$

   (b) **(5 points)** $\frac{x-1}{x+1} + \frac{x+1}{x-1}$. We find a common denominator:

$$\begin{aligned} \frac{x-1}{x+1} + \frac{x+1}{x-1} &= \frac{(x-1)(x-1)}{(x+1)(x-1)} + \frac{(x+1)(x+1)}{(x+1)(x-1)} \\ &= \frac{(x-1)^2 + (x+1)^2}{(x+1)(x-1)} \\ &= \frac{(x^2 - 2x + 1) + (x^2 + 2x + 1)}{x^2 - 1} \\ &= \frac{2x^2 + 2}{x^2 - 1} \end{aligned}$$

   If desired, the denominator may be left factored.

(c) **(5 points)** $\frac{(x^2y)^3}{y^{-1}}$. We use known exponential-rearrangement rules:

$$\begin{aligned}\frac{(x^2y)^3}{y^{-1}} &= \frac{x^{2\cdot 3}y^3}{y^{-1}} \\ &= x^6y^3 \cdot y^{-(-1)} = x^6y^4\end{aligned}$$

3. **(20 points)** *Find all real values satisfying the following inequalities, given either as conditions or as intervals:*

(a) **(5 points)** $-8r - 5 \leq -2$.

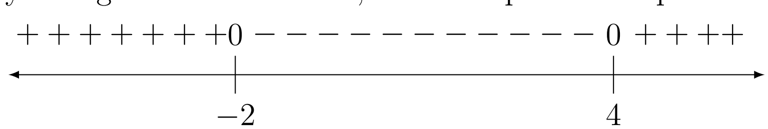
We add 5 to both sides to get $-8r \leq 3$; now we may divide both sides by -8, realizing that in doing so we reverse the inequality to get $r \geq \frac{-3}{8}$. THis could also be written in interval form as $[\frac{-3}{8}, \infty)$.

(b) **(7 points)** $x^2 - x - 8 > x$.

This is a quadratic inequality, so we subtract x from both sides to get a framing entirely in terms of sign: $x^2 - 2x - 8 > 0$; thus we want to explore the sign of the quadratic expression $x^2 - 2x - 8$.

Astute students might fasctor this as $(x - 4)(x + 2)$; alternatively, one might use the quadratic equation or completion of the square to find that $x^2 - 2x - 8$ is equal to zero when $x = 4$ or $x = -2$; in either case, we may begin building a number line to describe the sign:

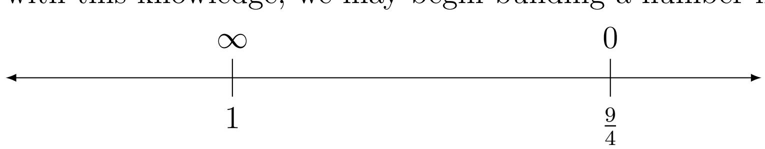
and now we need only probe in the untested intervals. To the left of -2, choosing $x = -3$ is a fine probe, yielding $(-3)^2 - 2(-3) - 8 = 7$, so to the left of -2 this expression is positive. Probing in between the zeroes, we can plug in $x = 0$ to get $0^2 - 2\cdot 0 - 8 = -8$, so this expression is negative between -2 and 4. Finally, to the right of 4, we look at $x = 5$, yielding $5^2 - 2\cdot 5 - 8 = 7$, so this expression is positive to the right of 4. Visually:



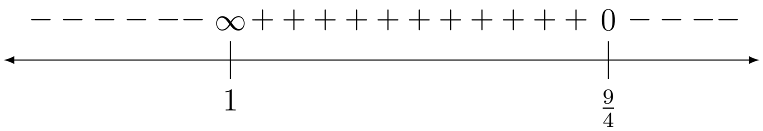
Since we are interested in the places where this expression is *positive*, there are two indicated intervals: $(-\infty, -2) \cup (4, \infty)$. Alternatively, you may assert that either $x < -2$ or $x > 4$.

(c) **(8 points)** $\frac{4x-9}{1-x} \leq 0$.

We want to explore the sign of the rational expression $\frac{4x-9}{1-x}$. We start by investigating the interesting points where this expression is zero or nonexistent. This expression is nonexistent when the denominator is zero, that is, when $1 - x = 0$, or when $x = 1$. The expression is zero when its numerator is zero, that is, when $4x - 9 = 0$ or $x = \frac{9}{4}$. Armed with this knowledge, we may begin building a number line to describe the sign:



and now we need only probe in the untested intervals. To the left of 1, choosing $x = 0$ is a fine probe, yielding $\frac{4 \cdot 0 - 9}{1-0} = -5$, so to the left of 0 this expression is negative. Probing in between the interesting points, we can plug in $x = 2$ to get $\frac{4 \cdot 2 - 9}{1-2} = 1$, so this expression is positive between 1 and $\frac{9}{4}$. Finally, to the right of $\frac{9}{4}$, we look at $x = 3$, yielding $\frac{4 \cdot 3 - 9}{1-3} = \frac{-3}{2}$, so this expression is negative to the right of $\frac{9}{4}$. Visually:



Since we are looking for the poitns where this expression is *zero or negative*, we see that there are two satisfactory intervals: $(-\infty, 1) \cup [\frac{9}{4}, \infty)$; alternatively, one could say that either $x < 1$ or $x \geq \frac{9}{4}$.

4. **(20 points)** *Find all real solutions to the following equations:*

(a) **(8 points)** $\frac{x}{x-3} = x + 1$.
Multiplying both sides by $x - 3$ and collecting terms to one side, we find that this equation is equivalent to:

$$\begin{aligned} \frac{x}{x-3} &= x + 1 \\ x &= (x+1)(x-3) \\ x &= x^2 - 3x + x - 3 \\ 0 &= x^2 - 3x - 3 \end{aligned}$$

which is a quadratic equation, and can be solved by either completion of the square or more directly via the quadratic formula:

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-3)}}{2} = \frac{3 \pm \sqrt{21}}{2}$$

(b) **(5 points)** $4(t - 2) = 12$.
We distribute the product on the left side and rearrange terms until the t stands alone:

$$\begin{aligned} 4t - 8 &= 12 \\ 4t &= 20 \\ t &= 5 \end{aligned}$$

(c) **(7 points)** $2u^2 + 5u - 3 = 4u^2 - u$.
Collecting terms on the right side, we can rearrange this into $0 = 2u^2 - 6u + 3$ (or a similar form on the left side). One may complete the square, or use the quadratic formula:

$$u = \frac{6 \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 2} = \frac{6 \pm \sqrt{12}}{4}$$

This may, but need not, be simplified to $\frac{3 \pm \sqrt{3}}{2}$.