1. **(20 points)** *Answer the following questions about optimization:*

   (a) **(8 points)** *The total cost of producing x units of a product is $C = 3x^2 - 90x + 1000$. Find the value of x for which the total cost is minimum.*

   This is a quadratic that opens upwards, so the minimum is at the vertex. We know the vertex of a quadratic $y = ax^2 + bx + c$ has x-coordinate of $\frac{-b}{2a}$, so in this case the minimum would appear at $\frac{-(-90)}{2 \cdot 3} = \frac{90}{6} = 15$.

   (b) **(12 points)** *There are 600 people who would attend a Deathmøle concert, if it were free. Every dollar charged for admission will reduce attendance by 24 people. What price for concert admission will maximize the total revenue?*

   Let us call our ticket price x. Then, the demand, as a function of ticket price, will be $600 - 24x$, since every dollar charged reduces our attendence by 24, from an initial attendence of 600 at a ticket price of \$0.

   If we sell $600 - 24x$ tickets, each at a price of x, our revenue will be $(600 - 24x) \cdot x = 600x - 24x^2$. We have the revenue as a quadratic function of price, so its maximum is its vertex; the quadratic $-24x^2 + 600x$ has vertex at $\frac{-600}{2(-24)} = \frac{600}{48} = 12.5$, so our optimal ticket price is \$12.50.

2. **(20 points)** *Answer the following questions involving the functions $f(x) = \frac{1}{x}$ and $g(x) = 2 - \sqrt{x+5}$.*

   (a) **(5 points)** *Find formulas for $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, and $\frac{f}{g}(x)$. Label which is which. You do not need to algebraically simplify your results.*

$$(f+g)(x) = \frac{1}{x} + (2 - \sqrt{x+5})$$
$$(f-g)(x) = \frac{1}{x} - (2 - \sqrt{x+5})$$
$$(fg)(x) = \frac{1}{x}(2 - \sqrt{x+5})$$
$$\frac{f}{g}(x) = \frac{\frac{1}{x}}{2 - \sqrt{x+5}}$$

   (b) **(5 points)** *Find formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$. Label which is which. You do not need to algebraically simplify your results.*

$$(f \circ g)(x) = f(g(x)) = f(2 - \sqrt{x+5}) = \frac{1}{2 - \sqrt{x+5}}$$
$$(g \circ f)(x) = g(f(x)) = g(\frac{1}{x}) = 2 - \sqrt{\frac{1}{x} + 5}$$

   (c) **(5 points)** *Determine the domain of $f(x)$ and the domain of $g(x)$. Label which is which.*

   $f(x)$ has a division, whose denominator must be nonzero; otherwise $f(x)$ is generally evaluatable, so $f(x)$'s domain contains all $x \neq 0$, or in interval form, $(-\infty, 0) \cup (0, \infty)$.

   $g(x)$ has a square root, whose argument cannot be negative; otherwise $g(x)$ is generally evaluatable, so $g(x)$ is evaluatable whenever $x + 5 \geq 0$; so its domain is given by $x \geq -5$, or in interval form, $[-5, \infty)$.

(d) **(5 points)** *Determine the domain of each of* $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, *and* $\frac{f}{g}(x)$. *Label which is which.*

The domains of $(f+g)(x)$, $(f-g)(x)$, and $(fg)(x)$ are all simply the overlap of the domains of $f(x)$ and $g(x)$, so their domains have the conditions $x \neq 0$ and $x \geq -5$ as seen in the previous section of this problem; in interval form, one could describe this as $[-5,0) \cup (0,\infty)$.

$\frac{f}{g}(x)$ starts from this same domain, but in addition we must exclude any values where $g(x)=0$. Since $g(x) = 2-\sqrt{x+5}$, we can solve for when $g(x)$=0:

$$\begin{aligned} 2-\sqrt{x+5} &= 0 \\ 2 &= \sqrt{x+5} \\ 4 &= x+5 \\ -1 &= x \end{aligned}$$

so the value $x=-1$ must be excluded from the domain of $\frac{f}{g}(x)$. Thus, $\frac{f}{g}(x)$ has domain defined by the conditions $x \geq -5$, $x \neq 0$, and $x \neq -1$; or, in interval form $[-5,-1) \cup (-1,0) \cup (0,\infty)$.

3. **(15 points)** *Find a function* $y = f(x)$ *fitting the description of each of the following graphs:*

(a) **(5 points)** *A line through* $(-2,1)$ *and* $(0,7)$.

It is easiest to find the slope, then find a line with that slope. The slope of this line is given by $\frac{7-1}{0-(-2)} = \frac{6}{2} = 3$. We thus want a line with slope 3 passing through $(0,7)$; that is, a line with slope 3 and y-intercept 7, which has slope-intercept form $y = 3x+7$.

(b) **(4 points)** *A line of slope* -2 *through the point* $(-1,7)$.

We can use point-slope form to get $(y-7) = -2(x-(-1))$; to isolate y on one side and get a proper function, we add 7 to both sides and get $y = -2(x+1)+7$, or, alternatively, $y = -2x+5$.

(c) **(6 points)** *A quadratic with vertex at* $(-3,-2)$ *which passes through the point* $(0,-8)$.

In standard form, a quadratic with vertex $(-3,-2)$ would be given by $y = a(x+3)^2-2$. We now only need the value of a, which we can get by knowing that $(0,-8)$ satisfies this function, so:

$$\begin{aligned} -8 &= a(0+3)^2-2 \\ -6 &= a \cdot 9 \\ \frac{-2}{3} = \frac{-6}{9} &= a \end{aligned}$$

so the final formula is $y = \frac{-2}{3}(x+3)^2-2$.

4. **(20 points)** *Answer the following questions about functions:*

(a) **(7 points)** *Write an equation for a function whose graph fits each of the given transformations of* $y=\sqrt{x}$:

i. **(2 points)** *The graph of* $f(x) = \sqrt{x}$ *is reflected over the* x*-axis.*
Such a transformation is $-f(x) = -\sqrt{x}$.

ii. **(2 points)** *The graph of* $f(x) = \sqrt{x}$ *is stretched vertically by a factor of 3.*
Such a transformation is $3f(x) = 3\sqrt{x}$.

iii. **(3 points)** *The graph of* $f(x) = \sqrt{x}$ *is shifted upwards by two units and right by four units.*
Such a transformation is $f(x-4)+2 = \sqrt{x-4}+2$.

(b) **(5 points)** *Determine the inverse function of* $g(x) = 5x - 3$.

The inverse is given by the rule $y = 5g^{-1}(y) - 3$; isolating the function proceeds via the following algebraic steps:

$$\begin{aligned} y &= 5g^{-1}(y) - 3 \\ y + 3 &= 5g^{-1}(y) \\ \frac{y+3}{5} &= g^{-1}(y) \end{aligned}$$

(c) **(7 points)** *Find the* x*-intercepts,* y*-intercept, and long-term behavior (or end behavior) of the function* $h(x) = (4x-2)(x+1)^2(x-4)$.

The y-intercept is given by $h(0) = (4 \cdot 0 - 2)(0+1)^2(0-4) = -2 \cdot 1^2(-4) = 8$.

The x-intercept is given by solving $h(x) = 0$; in order for $(4x-2)(x+1)^2(x-4)$ to be zero, one of those multiplicands must be zero; thus, $h(x) = 0$ if $4x - 2 = 0$ or $(x+1)^2 = 0$ or $x - 4 = 0$; these are true respectively when $x = \frac{1}{2}$, $x = -1$, or $x = 4$, so these three values are the x-intercepts.

Multiplying out this function gives $h(x) = 4x^4 + \cdots$; more fully it is $h(x) = 4x^4 - 10x^3 - 32x^2 - 2x + 8$, but those final terms are irrelevant. Since the leading term has positive coefficient (4), and even exponent (x^4), it has long-term behavior trending positive on both sides; i.e. arrows in the northeast and northwest corner of the graph.