1. **(17 points)** *Answer the following two questions.*

   (a) **(7 points)** *A colony of 100 mythrax bacteria grows so that after t hours, its population is given by the function $f(t) = 100e^{0.2t}$. How many hours will it take the colony to reach a size of 350 bacteria? You may leave unreduced exponents, natural logarithms, or base-10 logarithms in your answer.*

   This question, appropriately abstracted, is asking which value of $t$ causes $f(t)$ to equal 350. Since there is a formula for $f(t)$, this can be rephrased as an exponential equation:

$$\begin{aligned} 350 &= 100e^{0.2t} \\ \frac{350}{100} &= e^{0.2t} \\ \ln\frac{350}{100} &= \ln\left(e^{0.2t}\right) \\ \ln 3.5 &= 0.2t \\ \frac{\ln 3.5}{0.2} &= t \end{aligned}$$

   so the population is 350 in $\frac{\ln 3.5}{0.2}$ hours.

   (b) **(10 points)** *A zorkmid is a coin weighing 3 grams and worth 5 dollars; a quatloo is a coin weighing 4 grams worth 4 dollars. You have a pile of these two coins which weighs a total of 50 grams and which is worth 62 dollars. Construct and solve a system of two equations to find out how many there are of each coin.*

   Let us give the two quantities we seek names: let $z$ be the number of zorkmids, $q$ the number of quatloos. The total weight of the pile will then be $3z+4q$ grams, so $3z+4q = 50$; the total value of the pile is $5z+4q$ dollars, so $5z+4q = 62$, giving the system of equations

$$\begin{cases} 3z + 4q = 50 \\ 5z + 4q = 62 \end{cases}$$

   We can build the substitution $4q = 50 - 3z$ from the first equation, and substitute it into the second equation to get $5z + (50 - 3z) = 62$, or $2z = 12$, so $z = 6$. Putting that back into our substitution, $4q = 50 - 3 \cdot 6 = 32$, so $q = 8$. Thus, our pile contains 6 zorkmids and 8 quatloos.

2. **(21 points)** *Answer the following questions about exponential and logarithmic functions and equations:*

   (a) **(5 points)** *Evaluate each of the following exponential expressions; write your answer without exponents.*

   - $3^0$.
     $3^0 = 1$, since every positive number raised to the zeroth power is 1.
   - $27^{-2/3}$.
     We can unstack this into three separate procedures: raising to a negative power, which involves taking a reciprocal, raising to the fractional power $\frac{1}{3}$, which involves a radical, and an ordinary squaring.

$$27^{-2/3} = ((27^{1/3})^2)^{-1} = \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{3^2} = \frac{1}{9}$$

- $4^{-2}$.
  A negative power indicates a reciprocal, so $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$.
- $\left(\frac{2}{7}\right)^2$
  To exponentiate a fraction, we exponentiate both the numerator and denominator: $\left(\frac{2}{7}\right)^2 = \frac{2^2}{7^2} = \frac{4}{49}$.

(b) **(5 points)** *Evaluate each of the following logarithmic expressions; write your answer without logarithms.*

- $\log_9 3$.
  We know $3 = \sqrt{9}$, or in other words, $3 = 9^{1/2}$, so $\log_9 3 = \frac{1}{2}$.
- $\log_2 \frac{1}{8}$.
  We know $8 = 2^3$, so $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$. Thus, $\log_2 \frac{1}{8} = -3$
- $\log_2 16$.
  We know $16 = 2^4$, so $\log_2 16 = 4$.
- $\log_7 1$.
  The logarithm of 1 is always zero, regardless of base, since $a^0 = 1$ for all positive $a$.
- $\log_{100} 1000$
  We know $1000 = 10^3$ and $10 = \sqrt{100}$, so $1000 = (100^{1/2})^3 = 100^{3/2}$. Thus, $\log_{100} 1000 = \frac{3}{2}$.

(c) **(5 points)** *Condense the expression* $2 \ln x + \frac{1}{2} \ln(x^2 - 1) - 4 \ln 2$ *into a single logarithm.*

Using the sundry logarithm rules:

$$\begin{aligned} 2 \ln x + \frac{1}{2} \ln(x^2 - 1) - 4 \ln 2 &= \ln(x^2) + \ln(x^2 - 1)^{1/2} - \ln(2^4) \\ &= \ln \frac{x^2 (x^2 - 1)^{1/2}}{2^4} = \ln \frac{x^2 \sqrt{x^2 - 1}}{16} \end{aligned}$$

The last equality here was purely cosmetic, and isn't strictly necessary.

(d) **(7 points)** *Find a numeric solution to the equation* $\log_{10}(15x + 3) - \log_{10}(x + 1) = 1$. *Do not leave unsimplified exponents or logarithims in your answer.*

We collect the expression on the left into a single logarithm, and raise 10 to the power of each side to clear out the logarithm.

$$\begin{aligned} \log_{10}(15x + 3) - \log_{10}(x + 1) &= 1 \\ \log_{10} \frac{15x + 3}{x + 1} &= 1 \\ 10^{\log_{10} \frac{15x+3}{x+1}} &= 10^1 \\ \frac{15x + 3}{x + 1} &= 10 \\ 15x + 3 &= 10x + 10 \\ 5x &= 7 \\ x &= \frac{7}{5} \end{aligned}$$

3. **(17 points)** *Answer the following questions about polynomial and rational functions.*

(a) **(4 points)** *Find the possible rational roots of the polynomial $f(x) = 2x^3 - x^2 - 12x - 9$; you do not need to test to find which ones are actual zeroes.*

Since the leading term is 2, we must consider numbers whose denominators are 1 or 2; since the constant term is 9, we consider numbers whose numerators are 1, 3, or 9, and we also consider both positive and negative number, to get the range of possible roots:

$$\pm 1, \pm 3, \pm 9, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}$$

(b) **(5 points)** *Find the quotient and remainder when $4x^3 - 4x^2 + 3$ is divided by $2x^2 - x - 5$. Label which is which.*

Here is the division:

$$\begin{array}{r|l} & \quad\quad\quad\quad 2x - 1 \\ 2x^2 - x - 5 & \; 4x^3 - 4x^2 \quad\quad\quad + 3 \\ & -4x^3 + 2x^2 + 10x \\ \hline & \quad\quad -2x^2 + 10x + 3 \\ & \quad\quad\;\; 2x^2 \quad - x - 5 \\ \hline & \quad\quad\quad\quad\quad 9x - 2 \end{array}$$

Note this uses a slightly different scheme than we learned, in that each row is the negation of what we use and the rows are added rather than subtracted; unfortunately the typesetting package I am using for this purpose (the LaTeX package `polynom.sty`, for the idly curious) doesn't allow flexibility in this particular regard.

We can now extract the quotient $2x - 1$ and the remainder $9x - 2$.

(c) **(8 points)** *Find the $y$-intercept, $x$-intercepts (a.k.a. zeroes), vertical asymptotes, and horizontal/oblique asymptotes (a.k.a. long-term behavior) of the function $f(x) = \frac{(2x-1)(x+3)}{x^2-4}$. Label which is which, and if any of these features are completely absent, say so.*

The $y$-intercept is $f(0) = \frac{(0-1)(0+3)}{0-4} = \frac{3}{4}$.

The $x$=intercepts are given by solving for when the numerator of the rational function is zero, i.e. when $(2x - 1)(x + 3) = 0$, which occurs when $2x - 1 = 0$ or $x + 3 = 0$; in other words, at $x = \frac{1}{2}$ and $x = -3$.

The vertical asymptotes occur where the denominator is zero, i.e. when $x^2 - 4 = 0$, which happens when $x^2 = 4$ or $x = \pm 2$.

The long-term behavior of the function is given by dividing both the numerator and denominator by the largest degree appearing in the denominator ($x^2$ in this case) to get

$$f(x) = \frac{\frac{2x^2+5x-3}{x^2}}{\frac{x^2-4}{x^2}} = \frac{2 + \frac{5}{x} - \frac{3}{x^2}}{1 - \frac{4}{x^2}}$$

If we are looking at very large values of $x$, then these ractional terms are very close to zero, so the long-term appearance of the graph is given by

$$f(x) \approx \frac{2 + 0 - 0}{1 - 0} = 2$$

4. **(20 points)** *Find the solutions to the following systems of equations. Show your work.*

(a) **(6 points)** $\begin{cases} 2x + 3y = 3 \\ 3x - \phantom{3}y = 10 \end{cases}$

It is easy to express $y$ in terms of $x$ using the second equation so we shall do so: $y = 3x - 10$. Substituting that back into the first equation yields $2x + 3(3x - 10) = 3$, or $11x = 33$, so $x = 3$. Then $y = 3 \cdot 3 - 10 = -1$.

(b) **(6 points)** $\begin{cases} 4x + 5y + 2z = -3 \\ \phantom{4x+} 3y - \phantom{2}z = 14 \\ \phantom{4x + 5y} -3z = 15 \end{cases}$

This is a triangular system, so no rearrangement is necessary, just solution from the bottom up. The third equation gives us $z = -5$, which we can substitute into the second equation to get $3y + 5 = 14$, which can be solved to give $y = 3$. Finally, we substitute both of these pieces of knowledge into the first equation to get $4x + 5 \cdot 3 + 2(-5) = -3$, whose solution is $x = -2$.

(c) **(8 points)** $\begin{cases} x - 3y + 2z = 9 \\ 2x + 4y - 3z = -9 \\ 3x - 2y + 6z = 13 \end{cases}$

We can invoke a straightforward substitution using the first equation: $x = 9 + 3y - 2z$. Substiting that into the latter two gives:

$$\begin{cases} 2(9 + 3y - 2z) + 4y - 3z = -9 \\ 3(9 + 3y - 2z) - 2y + 6z = 13 \end{cases}$$

which, when distributed and rearranged, gives

$$\begin{cases} 10y - 7z = -27 \\ 7y \phantom{- 7z} = -14 \end{cases}$$

The latter equation gives us $y$ easily, so we sieze on the fact that $y = -2$. Then substituting into the former equation gives $10(-2) - 7z = -27$, which tells us that $z = 1$. Finally, our original substitution for $x$ can be evaluated to give $x = 9 + 3(-2) - 2 \cdot 1 = 1$.