1. **(20 points)** *Consider the vector-valued function* $\mathbf{r}(t) = \left\langle \frac{t^3}{3}, -2t, t^2 \right\rangle$.

   (a) **(5 points)** *Find the parametric equations of a tangent line to the curve described by* $\mathbf{r}(t)$ *at the point* $(9, -6, 9)$.

   We are looking at the curve at $t = 3$: necessary information will be the point of tangency (which we were given), and the orientation of the tangent line, which is given by $\mathbf{r}'(3)$. Since $\mathbf{r}'(t) = \langle t^2, -2, 2t\rangle$, in this particular case our orientation is $\langle 9, -2, 6\rangle$, so our parametric equation is

$$\begin{cases} x = 9t + 9 \\ y = -2t - 6 \\ z = 6t + 9 \end{cases}$$

   (b) **(5 points)** *Find the arclength along this curve from* $(0, 0, 0)$ *to* $(9, -6, 9)$.

   Above we determined $|\mathbf{r}'|$; now we calculate $|\mathbf{r}'(t)|$:

$$|\mathbf{r}'(t)| = \sqrt{(t^2)^2 + (-2)^2 + (2t)^2} = \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2 + 2)^2} = t^2 + 2$$

   and thus our arclength is

$$\int_0^3 (t^2 + 2)dt = \frac{1}{3}t^3 + 2t\Big]_0^3 = (9 + 6) - (0 + 0) = 15$$

   (c) **(5 points)** *Find the curvature of this curve at* $(9, -6, 9)$.

   While there are other formulas for the curvature, the easiest,e specially given our knowledge so far, is

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}.$$

   Note that $\mathbf{r}'(t) = \langle t^2, -2, 2t\rangle$ and $|\mathbf{r}'(t)| = t^2 + 2$ were calculated above; $\mathbf{r}''(t) = \langle 2t, 0, 2\rangle$ follows readily. Evaluating each of these at $t = 3$, we can find the curvature at the point $t = 3$ with ease:

$$\kappa = \frac{|\langle 9, -2, 6\rangle \times \langle 6, 0, 2\rangle|}{3^3} = \frac{|\langle -4, 18, 12\rangle|}{(3^2 + 2)^3} = \frac{2\sqrt{16 + 81 + 36}}{11^3} = \frac{2}{121}$$

   (d) **(5 points)** *Find the unit binormal vector at* $(9, -6, 9)$.

   Since we calculated $\mathbf{r}' \times \mathbf{r}''$ above, we already know a vector with the right orientation: $\langle 4, 2, 4\rangle$. Making it of unit length is a matter of calculating

$$\mathbf{B} = \frac{\langle -4, 18, 12\rangle}{|\langle -4, 18, 12\rangle|} = \left\langle \frac{-2}{11}, \frac{9}{11}, \frac{6}{11} \right\rangle$$

2. **(10 points)** *Consider the line given by the system of parametric equations* $x = 3t - 5$, $y = -t + 2$, $z = 4t + 4$, *and the plane given by the equation* $2x + 2y - z = 8$.

   (a) **(3 points)** *Does the line intersect the plane or not?*

We substitute the parametric equations for x, y, and z given by the line into the equation of the plane, and see if there is a t satisfying the equation:

$$\begin{aligned} 2(3t-5)+2(-t+2)-(4t+4) &= 8 \\ 6t-10-2t+4-4t-4 &= 8 \\ -10 &= 8 \end{aligned}$$

Since there is no value of t making this equation true (or if there is, I'd like to know about it!), there is no value of t such that the point $(3t-5, -t+2, 4t+4)$ lies in the plane $2x+2y-z=8$, so the line does not intersect the plane.

(b) **(7 points)** *If the line intersects the plane, determine the point where it does so; if it does not intersect the plane, determine the distance between the line and plane.*

Let us pick an arbitrary point on the line; we might use the $t=0$ point $(-5,2,4)$ for simplicity. Likewise pick an arbitrary point on the plane; there are several obvious choices, but in this particular solution, let's consider $(4,0,0)$. The vector $\mathbf{u}$ between these points is $\langle -9,2,4\rangle$. This vector includes significant components parallel to the line and plane, so it doesn't reflect the *shortest* distance between the line and plane; to find that, we need to find the length of the projection of $\mathbf{u}$ onto the line and plane's mutual normal. The plane's normal is clearly $\mathbf{n}=\langle 2,2,-1\rangle$; a cursory inspection indicates that it is perpendicular to the line as well (an aside: if the plane's normal were *not* perpendicular to the line, the line would necessarily intersect the plane, so we can take the fact that $\mathbf{n}$ is orthogonal to the line as given). Now we simply calculate the length of the normal component of $\mathbf{u}$:

$$\operatorname{comp}_{\mathbf{n}} \mathbf{u} = \frac{|\mathbf{n}\cdot\mathbf{u}|}{|\mathbf{n}|} = \frac{|2(-9)+2\cdot 2-1\cdot 4|}{\sqrt{2^2+2^2+(-1)^2}} = \frac{18}{3} = 6$$

3. **(15 points)** *Answer the following questions about surfaces in space:*

(a) **(5 points)** *Identify the surface described by the equation $z=x^2+3y^2-18y$ and state its orientation, if applicable.*

If we were to complete the square, we would find that $z+27=x^2+3(y^2-3)^2$. This is an elliptical paraboloid, which opens in the positive z-direction, since z does not appear squared in this expression. The vertex of the paraboloid (what might be thought of as the "center") is at $(0,3,-27)$, but the question did not in fact ask that.

(b) **(5 points)** *Determine (either by description or by sketching) the domain of the multivariable function $f(x,y)=\frac{\sqrt{1-x^2-y^2}}{x+y}$.*

There are two impediments to calculating $f(x,y)$: there is a fraction whose denominator might be zero and a square root whose argument might be negative. In order to be in the domain of the function, it thus must be the case that $x+y\neq 0$ and $1-x^2-y^2\geq 0$; More succinctly, $y\neq -x$ and $x^2+y^2\leq 1$.

Visually, this domain could be thought of as the unit disk (i.e. the boundary and interior of the unit circle), except for the line $y=-x$.

(c) **(5 points)** *Give a parametric system of equations describing the curve formed by the intersection of $z=x^2+3y^2-18y$ and $x+2y=4$.*

A curve satisfying both equations will probably allow us to express everything in terms of a single variable. The second equation presents an easy relationship: $x=4-2y$. Plugging that back into the first equation, $z = (4-2y)^2 + 3y^2 - 18y = 7y^2 - 34y + 16$. Thus, letting $y = t$, we have the parametric system

$$\begin{cases} x = 1 - 2t \\ y = t \\ z = 7t^2 - 34t + 16 \end{cases}$$

4. **(15 points)** *In the questions which follow,* $\mathbf{u} = 2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$ *and* $\mathbf{v} = \mathbf{i} - \mathbf{k}$.

   (a) **(4 points)** *Identify each of the following four expressions as a vector, a scalar, or as uncalculatable nonsense. You do not need to calculate these expressions or justify your assertions!*

      i. $(\mathbf{u} \cdot \mathbf{v}) - |\mathbf{v}|$.
      Working from the inside of each expression out: $\mathbf{u} \cdot \mathbf{v}$ is a scalar (dot product of two vectors), $|\mathbf{v}|$ is a scalar (length of a vector), and the difference of these two is a scalar minus a scalar (ordinary subtraction), which yields a scalar.

      ii. $\frac{\mathbf{u}-\mathbf{v}}{\mathbf{v}}$.
      Eyeballing this, we immediately see division by the vector $\mathbf{v}$, which is a nonsensical operation (there is no idea of what it means to divide by a vector).

      iii. $|\mathbf{u}|\mathbf{v} - (\mathbf{u} \times \mathbf{v})$.
      Working from the inside of each expression out: $|\mathbf{u}|$ is a scalar (length of a vector), so $|\mathbf{u}|\mathbf{v}$ is the product of a scalar and a vector, which is a vector (scalar-product of a vector). $\mathbf{u} \times \mathbf{v}$ is a vector (cross product of a vector), so the difference of the vectors $|\mathbf{u}|\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ (vector subtraction) is a vector.

      iv. $\frac{1}{|\mathbf{u}|}(\mathbf{u} \times \mathbf{v})$.
      Working from the inside of each expression out: $|\mathbf{u}|$ is a scalar (length of a vector), and $\frac{1}{|\mathbf{u}|}$ is its reciprocal, which is also a scalar. $\mathbf{u} \times \mathbf{v}$ is a vector (cross product of a vector), so the product of a scalar and a vector is the scalar-product of a vector, which is a vector.

   (b) **(3 points)** *Calculate* $\mathbf{u} \times \mathbf{v}$.
   Using any method you wish, one can find this to be $4\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$.

   (c) **(4 points)** *Find the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.
   The angle is given by $\theta = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{|u||v|}$, which we can calculate as a matter of routine:

$$\theta = \cos^{-1} \frac{2 \cdot 1 - 4 \cdot 0 + 6(-1)}{\sqrt{2^2 + 4^2 + 6^2}\sqrt{1^2 + 0^2 + 1^2}} = \cos^{-1} \frac{-4}{\sqrt{56}\sqrt{2}} = \cos^{-1} \frac{-1}{\sqrt{7}}$$

   (d) **(4 points)** *Calculate* $\text{proj}_{\mathbf{v}}\, \mathbf{u}$.

$$\text{proj}_{\mathbf{v}}\, \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{2 \cdot 1 - 4 \cdot 0 + 6(-1)}{1 \cdot 1 + 0 \cdot 0 + (-1)(-1)}\langle 1, 0, -1 \rangle = \langle -2, 0, 2 \rangle$$

5. **(5 point bonus)** *Prove on the back of this page that* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

There are astonishingly tedious arithmetical ways to do this one, since both sides will evaluate to $\langle a_2b_1c_2 + a_3b_1c_3 - a_2b_2c_1 - a_3b_3c_1, a_1b_2c_1 + a_3b_2c_3 - a_1b_1c_2 - a_3b_3c_2, a_1b_2c_1 + a_3b_2c_3 - a_1b_1c_2 - a_3b_3c_2 \rangle$. Some very interesting results can be developed without such recourse, however: we can immediately determine that both sides of the equation have the same *orientation*, for instance: we note that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ has an orientation characterized by its perpendicularity to both $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$; it is easy to show that $(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ satisfies both of these characteristics. First, we shall show that it is perpendicular to $\mathbf{b} \times \mathbf{c}$:

$$\begin{aligned} [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}] \cdot (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})[\mathbf{b} \cdot (\mathbf{b} \times c)] - (\mathbf{a} \cdot \mathbf{b})[\mathbf{c} \cdot (\mathbf{b} \times c)] \\ &= (\mathbf{a} \cdot \mathbf{c})0 - (\mathbf{a} \cdot \mathbf{b})0 = 0 \end{aligned}$$

We know that $\mathbf{b} \times \mathbf{c}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$; it is thus that we knew $\mathbf{b} \cdot (\mathbf{b} \times c)$ and $\mathbf{c} \cdot (\mathbf{b} \times c)$ were both zero. Now we do the same for perpendicularity to $\mathbf{a}$:

$$[(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}] \cdot \mathbf{a} = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{a}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{a}) = 0$$

and thus $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ have the same orientation. Showing that they have the same length is rather trickier and depends on a certain degree of unweildy arithmetic.