1. **(16 points)** *Write but do not evaluate the following integrals:*

   (a) **(6 points)** *A cylindrical integral to calculate the volume of the solid which lies in the first octant (where x, y, and z coordinates are all positive) under the paraboloid $z = 2 - x^2 - y^2$ and above the cone $z = \sqrt{x^2 + y^2}$.*

   The surface bounding this solid above can be written as $z = 2 - (x^2 + y^2) = 2 - r^2$; the lower bound is clearly $z = r$. These two solids intersect when $2 - r^2 = r$, which occurs when $r^2 + r - 2 = 0$, which is satisfied at the nonsensical value $r = -2$ and the eminently sensible value $r = 1$; thus, this surface's outermost edge is $r = 1$. The restriction to the first octant requires that $z \geq 0$, $x \geq 0$, and $y \geq 0$. The first of these restrictions is moot: the previous restriction $z \geq r$ ensures that z is non-negative; however, the restrictions on x and y constrain θ to those values whose cosines and sines are both positive, which is to say, the first-quadrant section $0 \leq \theta \leq \frac{\pi}{2}$.

   Our integral is thus:

   $$\iiint_E dV = \int_0^{\pi/2} \int_0^1 \int_r^{2-r^2} r dz dr d\theta$$

   (b) **(5 points)** *A polar integral to calculate $\iint_D e^{-x^2-y^2} dA$, where D is the region given by $x^2 + y^2 \leq 4$ with $y \geq 0$ and $x \leq y$.*

   The region in question consists of those values where $r \leq 2$, $\sin\theta \geq 0$, and $\cos\theta \leq \sin\theta$. The former constraint on θ we know to cover the first and second quadrants, which are the range $[0, \pi]$; however, for $0 \leq \theta < \frac{\pi}{4}$, θ's cosine exceeds its sine, so $x > y$; we must exclude this region as per the second *theta*-constraint, so $\frac{\pi}{4} \leq \theta \leq \pi$. Translating the integrand and multiplying by the integrating factor r, we get:

   $$\iint_D e^{-x^2-y^2} dA = \int_{\pi/4}^{\pi} \int_0^2 r e^{-r} dr d\theta$$

   (c) **(5 points)** *A spherical integral to calculate $\iiint_E x^2 + y^2 dV$ where E is the hollow hemi-spherical shell given by $1 \leq x^2 + y^2 + z^2 \leq 9$ with $y \geq 0$.*

   As seen above, $y \geq 0$ when $0 \leq \theta \leq \pi$; in addition, since $x^2 + y^2 + z^2 = \rho^2$, the first constraint is that $1 \leq \rho \leq 3$. ϕ is, in this case, unrestricted. Finally, the integrand $x^2 + y^2$ expands to $\rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) = \rho^2 \sin^2\phi$, and so:

   $$\iiint_E x^2 + y^2 dV = \int_0^{\pi} \int_0^{\pi} \int_1^3 (\rho^2 \sin^2\phi)(\rho^2 \sin\phi) d\rho d\phi d\theta$$

2. **(16 points)** *Calculate the following integrals, using whatever approach you find most effective:*

   (a) **(6 points)** *$\iiint_E 5y dV$ where E is the solid in the first octant bounded by the surfaces $x = 0$, $y = 0$, $y = x$, $z = x^2$, and $z = 4$.*

   The surfaces $z = x^2$ and $z = 4$ meet when $x^2 = 4$; in the first octant, this guarantees that the solid lies within $0 \leq x \leq 2$. The y=coordinate is bounded by 0 and x, so $0 \leq y \leq x$; finally, since the solid lies between $z = x^2$ and $z = 4$, $x^2 \leq z \leq 4$, and so we have an integral:

$$\iiint_E 5ydV = \int_0^2 \int_0^x \int_{x^2}^4 5ydzdydx$$
$$= \int_0^2 \int_0^x 5zy\Big]_{z=x^2}^{z=4} dydx = \int_0^2 \int_0^x 20y - 5x^2ydydx$$
$$= \int_0^2 10y^2 - \frac{5}{2}x^2y^2\Big]_{y=0}^{y=x} dx = \int_0^2 10x^2 - \frac{5}{2}x^4dx$$
$$= \frac{10}{3}x^3 - \frac{1}{2}x^5\Big]_0^2 = \frac{80}{3} - 16 = \frac{32}{3}$$

(b) **(6 points)** $\iint_D x - 2ydA$ *where D is the region bounded by the curves $y = 2 - x^2$ and $y = x^2$.*

These curves intersect when $2 - x^2 = x^2$, or where $x^2 = 1$, or at the points $x = \pm 1$, so the region is given by $-1 \leq x \leq 1$, and bounded above and below by $2 - x^2$ and x^2 respectively. Thus, our integral is:

$$\iint_D x - 2ydA = \int_{-1}^1 \int_{x^2}^{2-x^2} x - 2ydydx$$
$$= \int_{-1}^1 xy - y^2\Big]_{y=x^2}^{y=2-x^2} dx = \int_{-1}^1 2x - 2x^3 - (4 - 4x^2)dx$$
$$= x^2 - \frac{2}{4}x^4 - 4x + \frac{4}{3}x^3\Big]_{-1}^1 = (1-1) - \frac{1}{2}(1-1) - 4(1+1) + \frac{4}{3}(1+1) = \frac{-16}{3}$$

(c) **(4 points)** $\iint_D 2x + ydA$ *where D is the rectangle with corners $(-3, 0)$, $(2, 0)$, $(2, 2)$, and $(-3, 2)$.*

Since this is an integral over a rectangular region given by $-3 \leq x \leq 2$ and $0 \leq y \leq 2$, the integral can be set up as a simple iterated integral with constant limits:

$$\iint_D 2x + ydA = \int_{-3}^2 \int_0^2 2x + ydydx$$
$$= \int_{-3}^2 2xy + \frac{y^2}{2}\Big]_0^2 dx = \int_{-3}^2 4x + 2dx \quad = 2x^2 + 2x\Big]_{-3}^2 = 2(4-9) + 2(5) = 0$$

3. **(6 points)** *Using the transformations $x = 2u - v$ and $y = u + 4v$, evaluate $\iint_D x + ydA$ over the region D bounded by $4x + y = 18$, $4x + y = 27$, $2y - x = 0$, and $2y - x = 9$.*

Note that these four boundaries can be re-expressed easily in terms of u and v. $4x + y = 4(2u - v) + (u + 4v) = 9u$, so the inequality $18 \leq 4x + y \leq 27$ becomes $2 \leq u \leq 3$ with ease; likewise $2y - x = 2(u + 4v) - (2u - v) = 9v$, so $0 \leq 2y - x \leq 9$ becomes $0 \leq v \leq 1$. Now, we need only re-express the integrand and calculate the Jacobian. The integrand is $x + y = (2u - v) + (u + 4v) = 3u + 3v$; the Jacobian is calculated here:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = 9$$

so our integral can be transformed as follows:

$$\begin{aligned}\iint_D x+y\,dA &= \int_2^3\int_0^1 (3u+3v)\cdot 9\,dv\,du \\ &= \int_2^3 27uv+\frac{27}{2}v^2\Big]_{v=0}^{v=1} du = \int_2^3 27u+\frac{27}{2}du \\ &= \frac{27}{2}u^2+\frac{27}{2}u\Big]_2^3 = \frac{27}{2}(9+3-4-2) = \frac{27\cdot 6}{2} = 81\end{aligned}$$

4. **(8 points)** *Identify each of the following vector fields as either conservative or nonconservative; for each that is conservative, find a potential function:*

- $F(x,y) = (4x-3y+2)\mathbf{i}+(4y+1)\mathbf{j}$.
  We calculate $\frac{\partial}{\partial y}(4x-3y+2) = 3$, and $\frac{\partial}{\partial x}(4y+1) = 0$, so this vector field is nonconservative, since these results are nonequal.
- $G(x,y) = \langle \frac{2x}{y}+3, y^2-\frac{x^2}{y^2}\rangle$.
  We calculate $\frac{\partial}{\partial y}(\frac{2x}{y}+3) = \frac{-2x}{y^2}$, and $\frac{\partial}{\partial x}(y^2-\frac{x^2}{y^2}) = \frac{-2x}{y^2}$, so this vector field is conservative. Using the "partial integrals" of each component, we find that the potential function $g(x,y)$ is given by
  $$g(x,y) = \int \frac{2x}{y}+3dx = \frac{x^2}{y}+3x+C(y)$$
  and
  $$g(x,y) = \int y^2-\frac{x^2}{y^2}dy = \frac{y^3}{3}+\frac{x^2}{y}+D(x)$$
  Although these appear different, the term $3x$ in the first integral is represented in the second within the junk term $D(x)$, and likewise the term $\frac{y^3}{3}$ in the second integral is subsumed into the first integral's junk term $C(y)$. Thus, a potential function that matches both descriptions is $g(x,y) = \frac{x^2}{y}+3x+\frac{y^3}{3}$.
- $H(x,y) = \langle \ln y - e^x, 7\sin y + \frac{x}{y}\rangle$.
  We calculate $\frac{\partial}{\partial y}(\ln y - e^x) = \frac{1}{y}$, and $\frac{\partial}{\partial x}(7\sin y+\frac{x}{y}) = \frac{1}{y}$, so this vector field is conservative. Using the "partial integrals" of each component, we find that the potential function $h(x,y)$ is given by
  $$h(x,y) = \int \ln y - e^x dx = x\ln y - e^x + C(y)$$
  and
  $$h(x,y) = \int 7\sin y + \frac{x}{y}dy = -7\cos y + x\ln y + D(x)$$
  Although these appear different, the term $-e^x$ in the first integral is represented in the second within the junk term $D(x)$, and likewise the term $-7\cos y$ in the second integral is subsumed into the first integral's junk term $C(y)$. Thus, a potential function that matches both descriptions is $h(x,y) = x\ln y - 7\cos y - e^x$.

5. **(14 points)** *Calculate the following path integrals.*

(a) **(5 points)** *$\int_C x^2 ds$ where C is the line segment from $(0,4)$ to $(3,2)$.*
Let us parameterize this segment with $x = 3t$, $y = 4-2t$, for $0 \le t \le 1$. Then this integral is:

$$\begin{aligned}\int_C x^2 ds &= \int_0^1 x(t)^2\sqrt{x'(t)^2 + y'(t)^2}dt \\ &= \int_0^1 9t^2\sqrt{3^2+(-2)^2}dt \\ &= 3t^3\sqrt{13}\Big]_0^1 = 3\sqrt{13}\end{aligned}$$

(b) **(5 points)** *$\int_C F \cdot d\mathbf{r}$, where $F(x,y,z) = \langle 4y+z, 3x-z, 2z\rangle$ and C is the curve given by $x = t$, $y = t^2$, and $z = t$ from $(0,0,0)$ to $(2,4,2)$.*
Since a parameterization is already given (and the curve in question is from $t = 0$ to $t = 2$, we may evaluate this directly:

$$\begin{aligned}\int_C F \cdot d\mathbf{r} &= \int_0^2 \langle 4y(t)+z(t), 3x(t)-z(t), 2z(t)\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle dt \\ &= \int_0^2 \langle 4t^2+t, 2t, 2t\rangle \cdot \langle 1, 2t, 1\rangle dt \\ &= \int_0^2 8t^2 + 3tdt \\ &= \frac{8}{3}t^3 + \frac{3}{2}t^2\Big]_0^2 = \frac{64}{3} + 6 = \frac{82}{3}\end{aligned}$$

(c) **(4 points)** *$\int_C 2x - 3ydy$ where C is the curve $y = x^2 - 1$ from $(1,0)$ to $(4,15)$.*
Using the parameterization $x = t$, $y = t^2 - 1$ on $1 \le t \le 4$, this integral becomes:

$$\begin{aligned}\int_C 2x - 3ydy &= \int_1^4 [2x(t) - 3y(t)]y'(t)dt \\ &= \int_1^4 4t^2 - 6t^3 + 6tdt \\ &= \frac{4}{3}t^3 - \frac{3}{2}t^4 + 3t^2\Big]_1^4 \\ &= \frac{4\cdot 63}{3} - \frac{3\cdot 255}{2} + 3\cdot 15 = \frac{-507}{2}\end{aligned}$$

6. **(5 point bonus)** *On the back of this sheet, identify the shape of the solid whose volume is described by the integral $\int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_0^{3\csc\phi} \rho^2\sin\phi d\rho d\phi d\theta$, and calculate its volume without taking an integral.*

This integral is obviously $\iiint_E dV$, or the volume of E, for E given by $0 \le \theta \le 2\pi$, $\frac{\pi}{4} \le \phi \le \frac{3\pi}{4}$, and $0 \le \rho \le 3\csc\phi$. The shape has a full range of θ-values and no dependencies of the other parameters on θ, so it can be expected to be radially symmetric about the z-axis.

Considering the ϕ-bounds, which are constant, we can expect this solid to be bounded between two cones. When $\phi = \frac{\pi}{4}$, we know that z is positive (since $\cos\frac{\pi}{4}$ is positive, and $\frac{\sqrt{x^2+y^2}}{z} =$

$\tan\frac{\pi}{4} = 1$, so the half-cone $z = \sqrt{x^2+y^2}$ describes the surface $\phi = \frac{\pi}{4}$; likewise, when $\phi = \frac{\pi}{4}$, $z = -\sqrt{x^2+y^2}$, so two bounds on the solid E are the cones $z = \sqrt{x^2+y^2}$ and $z = -\sqrt{x^2+y^2}$.

Now we consider the ρ-bound $\rho \leq 3\csc\phi$. Multiplying both sides by $\sin\phi$ (which is guaranteed to be positive), we have $\rho\sin\phi \leq 3$; note that $\rho\sin\phi = \sqrt{x^2+y^2}$, so this bound is the same as saying that our solid lies inside the cylinder $\sqrt{x^2+y^2} = 3$.

So, our solid is a cylinder of radius 3 bounded above and below by right cones whose radius equals their height and which meet at the origin. We can thus see that this figure is a cylinder of height 6, with two cones of height 3 and radius 3 carved out. We may calculate this volume to be:

$$6(\pi\cdot 3^2) - 2\cdot\frac{1}{3}\cdot 3(\pi\cdot 3^2) = 36\pi$$