1. **(15 points)** Consider the vector-valued function $\mathbf{r}(t) = \langle 2\sin t, 6t, 2\cos t\rangle$

   (a) **(5 points)** Find the parametric equations of a tangent line to the curve described by $\mathbf{r}(t)$ at the point $(0, 0, 2)$.

   (b) **(5 points)** Find the arclength along this curve from $(0, 0, 2)$ to $(0, 6\pi, -2)$.

   (c) **(5 points)** Find the unit binormal vector at $(\sqrt{3}, 2\pi, 1)$.

2. **(10 points)** Answer the following questions:

   (a) **(5 points)** Given the trajectory $\mathbf{r}(t) = t^3\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$, identify the tangential and normal components of the acceleration vector when $t = 1$.

   (b) **(5 points)** Find an equation of the tangent plane to the surface $2xy - 6y^2 + z - xz^2 = -3$ at the point $(2, 1, 1)$.

3. **(10 points)** Consider the line given by the system of parametric equations $x = 1+t$, $y = 2-t$, $z = 3t$, and the line given by the system $x = 2 - s$, $y = 1 + 2s$, $z = 4 + s$.

   (a) **(3 points)** Do the lines intersect or not?

   (b) **(7 points)** If the lines intersect, determine the point where they do so; if they do not intersect, determine the distance between the two lines.

4. **(15 points)** In the questions which follow, $\mathbf{u} = 1\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} + 4\mathbf{k}$.

   (a) **(4 points)** Identify each of the following four expressions as a vector, a scalar, or as uncalculatable nonsense. You do not need to calculate these expressions or justify your assertions!

      i. $|\mathbf{u}| - (\mathbf{u} \cdot \mathbf{v})$.
      ii. $(\mathbf{u} - \mathbf{v}) \times (\mathbf{u} + \mathbf{v})$.
      iii. $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{u}$.
      iv. $|\mathbf{v}|(\mathbf{u} \times \mathbf{v})$.

   (b) **(3 points)** Calculate $\mathbf{u} \times \mathbf{v}$.

   (c) **(4 points)** Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (d) **(4 points)** Calculate $\text{proj}_{\mathbf{v}}\, \mathbf{u}$.

5. **(15 points)** Answer the following questions about optimization:

   (a) **(7 points)** Find the critical points of $f(x, y) = x^3y + 12x^2 - 8y$ and identify each as a local maximum, local minimum, or saddle point.

   (b) **(8 points)** Find positive x, y, and z such that $x + y + z = 1$ and which maximize the value of x^2y^3z.

6. **(10 points)** Calculate the following integrals:

   (a) **(4 points)** Calculate $\iint_A 3xy^2 dA$ if A is the rectangle $2 \leq x \leq 4, -1 \leq y \leq 1$.

   (b) **(6 points)** Find the integral of $x + y$ over the region bounded by the curves $y = x$ and $y = x^4$.

7. **(15 points)** Set up *but do not evaluate* the following integrals.

   (a) **(5 points)** An integral to determine the volume of the solid enclosed by the planes $z = 0$, $x = 0$, $y = 4$, $y = 2x$, and $z = 1 + y$.

   (b) **(5 points)** A cylindrical form for the integral $\iiint_E (x^3 + xy^2)dV$ over the solid in the first octant lying beneath the paraboloid $z = 1 - x^2 - y^2$.

   (c) **(5 points)** A spherical integral to calculate the integral $\iiint_E (x^2 + y^2)dV$, where E is the hemisphere lying above the xy-plane and below the sphere $x^2 + y^2 + z^2 = 1$.

8. **(10 points)** Calculate the following path integrals:

   (a) **(5 points)** $\int_C x^2yds$ where C is the line segment from $(0, 6)$ to $(2, 0)$.

   (b) **(5 points)** $\int_C F \cdot d\mathbf{r}$, where $F(x, y, z) = \langle yz, xz, xy\rangle$ and C is the curve given by $x = t$, $y = t^2$, and $z = t^3$ from $(0, 0, 0)$ to $(2, 4, 8)$.

9. **(15 points)** Answer the following vector-field-related questions:

   (a) **(4 points)** Determine whether the vector field $G(x,y) = (x^3 + 4xy)\mathbf{i} + (2x^2 - y^3)\mathbf{j}$ is conservative; if conservative, find a potential function.

   (b) **(7 points)** Use Green's Theorem to evaluate $\int_C x^2 dx + 2xydy$, where C is the path consisting of a parabolic arc $y = x^2$ from $(0,0)$ to $(2,4)$, together with straight-line paths from $(2,4)$ to $(0,4)$ and from $(0,4)$ to $(0,0)$.

   (c) **(4 points)** Find the divergence and curl of the vector field $\mathbf{F}(x,y,z) = xyz\mathbf{i} - x^2y\mathbf{k}$

10. **(5 point bonus)** Let a and b be constants that are both nonzero, and let $f(x,y)$ be subject to the condition

$$f(x,y) = a\frac{\partial}{\partial x}f(x,y) + b\frac{\partial}{\partial y}f_y(x,y)$$

Prove that, if there is some constant M such that $|f(x,y)| \leq M$ everywhere, then $f(x,y) = 0$ everywhere. (Note: this problem appeared on the 2010 Putnam Exam)