1. **(10 points)** *Consider the two lines given by the system of parametric equations $x = t + 2$, $y = 4t - 1$, $z = -5t$ and the system of parametric equations $x = 5s - 1$, $y = 3s$, $z = -s + 3$.*

   (a) **(3 points)** *Are the lines parallel, intersecting, or skew?*

   The orientation of the first line is given by the vector $\langle 1, 4, -5\rangle$; the orientation of the second is given by $\langle 5, 3, -1\rangle$. These are not scalar multiples of each other, hence the vectors and their associated lines are nonparallel.

   To test if they intersect, let us consider equality between the parametric equations, so that $t + 2 = 5s - 1$, $4t - 1 = 3s$, and $-5t = -s + 3$. The first equation yields $t = 5s - 3$. If we substitute that into the second equation we get $4(5s - 3) - 1 = 3s$, or $s = \frac{13}{17}$, while substituting it into the third gives $-5(5s-3) = -s+3$, or $s = \frac{-6}{24}$. Since no single value of s satisfies both equations, these three equations have no simultaneous solution and these lines do not intersect (alternatively: solve part (b) and note that the answer is nonzero).

   (b) **(7 points)** *If the lines intersect, determine the point where it does so; if they do not intersect, determine the distance between the lines.*

   Let us choose arbitrary points A and B on the two diistinct lines; plugging in $t = 0$ and $s = 0$ gives us the easily calculated points $(2, -1, 0)$ and $(-1, 0, 3)$. The vector between these points is $\mathbf{u} = \langle -3, 1, 3\rangle$; however, to find a *shortest* vector between the two lines, we want to find the component of this vector which is orthogonal to both lines, namely that component parallel to

$$\mathbf{n} = \langle 1, 4, -5\rangle \times \langle 5, 3, -1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -5 \\ 5 & 3 & -1 \end{vmatrix} = \langle 11, -24, -17\rangle$$

   So we find

$$|\operatorname{comp}_{\mathbf{n}} \mathbf{u}| = \frac{|\mathbf{u} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|-33 - 24 - 51|}{\sqrt{11^2 + 24^2 + 17^2}} = \frac{108}{\sqrt{986}}$$

2. **(15 points)** *Answer the following questions about surfaces in space:*

   (a) **(5 points)** *Determine (either by description or by sketching) the domain of the multivariable function $f(x, y) = \frac{x^3 + \ln y}{\sqrt{x-y}}$.*

   There are three impediments to calculating $f(x, y)$: there is a fraction whose denominator might be zero, a square root whose argument might be negative, and a natural logarithm whose argument might be nonpositive. In order to be in the domain of the function, it thus must be the case that $\sqrt{x - y} \neq 0$, $x - y \geq 0$, and $y > 0$; the first two could be combined so that the condition is that $x - y > 0$ and $y > 0$ (or, even more succinctly, $x > y > 0$.

   Visually, this domain could be thought of as the lower half of the first quadrant (i.e. everything between "due east" and "northeast").

   (b) **(5 points)** *Identify the surface described by the equation $y = 2x^2 - z^2 + 4z$ and state its orientation, if applicable.*

   If we were to complete the square, we would find that $y - 4 = 2x^2 - (z - 2)^2$. This is a hyperbolic paraboloid, a.k.a. "saddle surface", oriented in the y-direction, since y does not appear squared in this expression. The middle of the saddle (what might be thought of as the "center") is at $(0, 4, 2)$, but the question did not in fact ask that.

(c) **(5 points)** *Give a parametric system of equations describing the curve formed by the intersection of* $y = 2x^2 - z^2 + 4z$ *and* $4x + z = 1$.

A curve satisfying both equations will probably allow us to express everything in terms of a single variable. The second equation presents an easy relationship: $z = 1 - 4x$. Plugging that back into the first equation, $y = 2x^2 - (1-4x)^2 + 4(1-4x) = -14x^2 - 8x + 3$. Thus, letting $x = t$, we have the parametric system

$$\begin{cases} x = t \\ y = -14t^2 - 8t + 3 \\ z = 1 - 4t \end{cases}$$

3. **(20 points)** *Consider the vector-valued function* $\mathbf{r}(t) = \left\langle t^2, \frac{2}{3}t^3, t \right\rangle$.

(a) **(5 points)** *Find the parametric equations of a tangent line to the curve described by* $\mathbf{r}(t)$ *at the point* $(1, -\frac{2}{3}, -1)$.

We are looking at the curve at $t = -1$: necessary information will be the point of tangency (which we were given), and the orientation of the tangent line, which is given by $\mathbf{r}'(-1)$. Since $\mathbf{r}'(t) = \langle 2t, 2t^2, 1\rangle$, in this particular case our orientation is $\langle -2, 2, 1\rangle$, so our parametric equation is

$$\begin{cases} x = -2t + 1 \\ y = 2t - \dfrac{2}{3} \\ z = t - 1 \end{cases}$$

(b) **(5 points)** *Find the arclength along this curve from* $(0, 0, 0)$ *to* $(1, -\frac{2}{3}, -1)$.

Above we determined $|\mathbf{r}'|$; now we calculate $|\mathbf{r}'(t)|$:

$$|\mathbf{r}'(t)| = \sqrt{(2t)^2 + (2t^2)^2 + 1^2} = \sqrt{4t^4 + 4t^2 + 1} = \sqrt{(2t^2+1)^2} = 2t^2 + 1$$

and thus our arclength is

$$\int_{-1}^{0} (2t^2 + 1)dt = \frac{2}{3}t^3 + t\Big]_{-1}^{0} = (0 + 0) - (-\frac{2}{3} - 1) = \frac{5}{3}$$

(c) **(5 points)** *Find the curvature of this curve at* $(1, -\frac{2}{3}, -1)$.

While there are other formulas for the curvature, the easiest,e specially given our knowledge so far, is

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}.$$

Note that $\mathbf{r}'(t) = \langle 2t, 2t^2, 1\rangle$ and $|\mathbf{r}'(t)| = 2t^2 + 1$ were calculated above; $\mathbf{r}''(t) = \langle 2, 4t, 0\rangle$ follows readily. Evaluating each of these at $t = -1$, we can find the curvature at the point $t = -1$ with ease:

$$\kappa = \frac{|\langle -2, 2, 1\rangle \times \langle 2, -4, 0\rangle|}{3^3} = \frac{|\langle 4, 2, 4\rangle|}{27} = \frac{\sqrt{16 + 4 + 16}}{27} = \frac{2}{9}$$

(d) **(5 points)** *Find the unit binormal vector at* $(1, -\frac{2}{3}, -1)$.

Since we calculated $\mathbf{r}' \times \mathbf{r}''$ above, we already know a vector with the right orientation: $\langle 4, 2, 4\rangle$. Making it of unit length is a matter of calculating

$$\mathbf{B} = \frac{\langle 4, 2, 4\rangle}{|\langle 4, 2, 4\rangle|} = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$$

4. **(15 points)** *In the questions which follow,* $\mathbf{u} = \langle 3, -1, 2\rangle$ *and* $\mathbf{v} = \langle 0, 3, 4\rangle$.

(a) **(4 points)** *Find the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

The angle is given by $\theta = \cos^{-1} \frac{\mathbf{u}\cdot\mathbf{v}}{|u||v|}$, which we can calculate as a matter of routine:

$$\theta = \cos^{-1} \frac{3 \cdot 0 - 1 \cdot 3 + 2 \cdot 4}{\sqrt{3^2 + 1^2 + 2^2}\sqrt{0^2 + 3^2 + 4^2}} = \cos^{-1} \frac{1}{\sqrt{14}}$$

(b) **(3 points)** *Calculate* $\mathbf{u} \times \mathbf{v}$.

Using any method you wish, one can find this to be $-10\mathbf{i} - 12\mathbf{j} + 9\mathbf{k}$.

(c) **(4 points)** *Identify each of the following four expressions as a vector, a scalar, or as uncalculatable nonsense. You do not need to calculate these expressions or justify your assertions!*

i. $(\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \times \mathbf{v})$.
Working from the inside of each expression out: $\mathbf{u} \cdot \mathbf{v}$ is a scalar (dot product of two vectors), $\mathbf{u} \times \mathbf{v}$ is a vector (cross product of two vectors), and the product of these two is a scalar times a vector (scalar product), which yields a vector.

ii. $|\mathbf{u}|^2 - (\mathbf{v} \cdot \mathbf{v})$.
Working from the inside of each expression out: $\mathbf{v} \cdot \mathbf{v}$ is a scalar (dot product of two vectors), $|\mathbf{u}|$ is a scalar and so is its square, and the difference of these two is a scalar minus a scalar (conventional subtraction), which yields a scalar.

iii. $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$.
Working from the inside of each expression out: $\mathbf{u}+\mathbf{v}$ is a vector (sum of two vectors), $\mathbf{u}-\mathbf{v}$ is a vector (difference of two vectors), and the cross product of these two vectors is a vector.

iv. $\mathbf{u} + (\mathbf{u} \cdot \mathbf{v})$.
Working from the inside of each expression out: $\mathbf{u} \cdot \mathbf{v}$ is a scalar (dot product of two vectors), $\mathbf{u}$ is a vector, and the sum of these two, which are a scalar and a vector, is a nonsensical operation.

(d) **(4 points)** *Calculate* $\text{proj}_{\mathbf{v}}\, \mathbf{u}$.

$$\text{proj}_{\mathbf{v}}\, \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{3 \cdot 0 - 1 \cdot 3 + 2 \cdot 4}{0 \cdot 0 + 3 \cdot 3 + 4 \cdot 4}\langle 0, 3, 4\rangle = \left\langle 0, \frac{3}{5}, \frac{4}{5} \right\rangle$$