1. **(12 points)** *Answer the following questions about the function* $f(x, y) = \ln(xy - 3y)$.

   (a) **(4 points)** *Find all the partial derivatives of* f. *Label which is which.*
   Using the chain rule, $f_x(x, y) = \frac{y}{xy-3y} = \frac{1}{x-3}$; likewise, $f_y(x, y) = \frac{x-3}{xy-3y} = \frac{1}{y}$.

   (b) **(4 points)** *Find all the second partial derivatives of* f.
   It is easy to determine $f_{xx}(x, y) = \frac{\partial}{\partial x}\frac{1}{x-3} = \frac{-1}{(x-3)^2}$; a similar route will show that $f_{yy}(x, y) = \frac{-1}{y^2}$. Finally, taking either $\frac{\partial}{\partial x} f_y(x, y)$ or $\frac{\partial}{\partial y} f_x(x, y)$ will give zero, so $f_{xy}(x, y) = 0$.

   (c) **(4 points)** *Find an equation of the tangent plane to* $z = \ln(xy - 3y)$ *at the point* $(4, 1, 0)$.
   The general template for the tangent plane at a point (x_0, y_0, z_0) of a curve $z = f(x, y)$ is given by

   $$(z - z_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

   $f_x(4, 1) = \frac{1}{4-3} = 1$; likewise $f_y(4, 1) = \frac{1}{1} = 1$, so the equation of the tangent plane in this particular case is

   $$z = 0 = 1(x - 4) + 1(y - 1)$$

   or alternatively simply $z = x + y - 5$.

2. **(10 points)** *Answer the following questions aboud the function* $f(x, y, z) = \frac{2xy-4x^2}{z^2-1}$.

   (a) **(5 points)** *Where* $u = \frac{2xy-4x^2}{z^2-1}$, *calculate* $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, *and* $\frac{\partial u}{\partial z}$.
   Despite this being a fraction, we rarely have to use the quotient rule; since the denominator contains neither x nor y, in two of the partials we may treat the division by $z^2 - 1$ as if it were a constant multiplication:

   $$\frac{\partial u}{\partial x} = \frac{2y - 8x}{z^2 - 1}$$
   $$\frac{\partial u}{\partial y} = \frac{2x}{z^2 - 1}$$

   In the case of the derivative with respect to z, we must use either the quotient rule or the chain rule, to get:

   $$\frac{\partial u}{\partial z} = \frac{-2z(2xy - 4x^2)}{(z^2 - 1)^2}$$

   (b) **(5 points)** *Find an equation of the tangent plane to the level surface* $\frac{2xy-4x^2}{z^2-1} = 2$ *at the point* $(1, 5, -2)$.
   The tangent plane to a level surface $f(x, y, z) = k$ at a point (x_0, y_0, z_0) is given by the formula

   $$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

   The formulas for f_x, f_y, and f_z were calculated above, so all we need to do now is plug in $(1, 5, -2)$: $f_x(1, 5, -2) = \frac{2\cdot5-8\cdot1}{(-2)^2-1} = \frac{2}{3}$, $f_y(1, 5, -2) = \frac{2\cdot1}{(-2)^2-1} = \frac{2}{3}$, and $f_z(1, 5, -2) = \frac{-2(-2)(2\cdot1\cdot5-4\cdot1^2)}{[(-2)^2-1]^2} = \frac{24}{9} = \frac{8}{3}$, giving the equation of the tangent plane to be:

   $$\frac{2}{3}(x - 1) + \frac{2}{3}(y - 5) + \frac{8}{3}(z + 2) = 0$$

   Which can be beautified, if desired, to $x + y + 4z + 2 = 0$, or something similar.

3. **(8 points)** *Answer the following questions.*

   (a) **(4 points)** *Given* $x = \ln(2s - t)$, $y = \arcsin t$, *and* $u = x^2 + y^2$, *calculate* $\frac{\partial u}{\partial s}$ *and* $\frac{\partial u}{\partial t}$*; your answers need not be algebraically simplified.*

   Note that u is a function of x and y, x is a function of s and t, and y is a function solely of t, so by the chain rule:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} = 2x\frac{2}{2s - t}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = 2x\frac{-1}{2s - t} + 2y\frac{1}{\sqrt{1 - t^2}}$$

   (b) **(4 points)** *Calculate* $\iint_A (x^2 - 4xy)dA$ *if* A *is the rectangle* $1 \le x \le 4, 2 \le y \le 3$.

   Using iterated integrals:

$$\begin{aligned}\iint_A (x^2 - 4xy)dA &= \int_1^4 \int_2^3 x^2 - 4xydydx \\ &= \int_1^4 yx^2 - 2xy^2\Big]_{y=2}^{y=3} dx \\ &= \int_1^4 3x^2 - 18x - 2x^2 + 8xdx = \int_1^4 x^2 - 10xdx \quad = \frac{1}{3}x^3 - 5x^2\Big]_1^4 = \frac{64}{3} - 80 -\end{aligned}$$

4. **(15 points)** *Answer the following questions about the rates of change of the function* $f(x, y) = xy^2 - 2x^2 + 4y - 7$ *at the point* $(2, -1)$.

   (a) **(5 points)** *Find the derivative of* $f(x, y) = xy^2 - 2x^2 + 4y - 7$ *from the point* $(2, -1)$ *in the direction* $\langle 3, -4\rangle$.

   Let us calculate $\nabla f(2, -1)$; this will be useful to us through all the remaining questions:

$$\nabla f(x, y) = \langle y^2 - 4x, 2xy + 4\rangle$$

   so $\nabla f(2, -1) = \langle -7, 0\rangle$. Given a direction vector $\mathbf{u}$, finding the rate of change in that direction is straightforward once we have the gradient:

$$D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{-7 \cdot 3 + 0(-4)}{\sqrt{3^2 + (-4)^2}} = \frac{-21}{5}$$

   (b) **(5 points)** *Find the direction in which the function* $f(x, y) = xy^2 - 2x^2 + 4y - 7$ *has the greatest rate of change at the point* $(2, -1)$*, and identify its rate of change in that direction.*

   The maximum change occurs in the direction $\nabla f(2, -1)$, which is $\langle -7, 0\rangle$ (or, to give a unit direction, $\langle -1, 0\rangle$), and the rate of change in this direction is $|\nabla f(2, -1)| = 7$.

   (c) **(5 points)** *Identify a direction in which* $f(x, y) = xy^2 - 2x^2 + 4y - 7$ *has a zero rate of change at the point* $(2, -1)$.

   In order for the rate of change in a direction $\mathbf{u}$ to be zero, $\mathbf{u}$ must be perpendicular to $\nabla f(2, -1)$. One can solve the dot product $\langle -7, 0\rangle \cdot \langle x, y\rangle = 0$, or can simply note that the perpendicular to a horizontal vector is vertical, so any direction vector $\langle 0, k\rangle$ suffices.

5. **(15 points)** *Answer the following optimization-related problems.*

(a) **(7 points)** *Find the critical points of $f(x, y) = 2x^3 - xy^2 + 5x^2 + 4x$, and identify their type.*

Note $\nabla f = \langle 6x^2 - y^2 + 10x + 4, -2xy\rangle$. At a critical point, this must be $\mathbf{0}$, so critical points satisfy the pair of equations

$$\begin{cases} 6x^2 - y^2 + 10x + 4 = 0 \\ -2xy = 0 \end{cases}$$

The second condition is obviosuly satisfied when either $x = 0$ or $y = 0$, so let us address the first equation in light of those two possibilities:

If $x = 0$, the first equation becomes $-y^2 + 4 = 0$, so $y = \pm 2$. Thus, we have the critical points $(0, -2)$ and $(0, 2)$.

If $y = 0$, the first eqaution becomes $6x^2 + 10x + 4$; using the quadratic formula or factoring, this equation can be seen to have roots $x = -1$ and $x = -\frac{2}{3}$, so we have critical points $(-1, 0)$ and $(-\frac{2}{3}, 0)$.

Now, we need to identify the critical points by type by evaluating

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (12x + 10)(-2x) - 4y^2$$

When $x = 0$, this is clearly nonpositive; specifically at $(0, -2)$ and $(0, 2)$ this evaluates to -16 so both of those are saddle points. Likewise at $(-1, 0)$, D evalautes to -4, so this point too is a saddle point. But at $(-\frac{2}{3}, 0)$, D evaluates to $\frac{8}{3}$, so this is an extremum; since $f_{xx}(x, y) = 12x + 10 = 2 > 0$, this is specifically a minimum.

(b) **(8 points)** *Find the three positive numbers x, y, and z whose sum is 100 which maximize the value of x^2yz.*

This is a constrained optimization problem; the constraint if $x + y + z = 100$, so we will define $g(x, y, z) = x + y + z$; the goal function is $f(x, y, z) = x^2yz$. To solve this we will use Lagrange multipliers:

$$\nabla f = \lambda \nabla g$$

which, for these specific values of f and g, gives

$$\langle 2xyz, x^2z, x^2y\rangle = \lambda\langle 1, 1, 1\rangle$$

which, together with our constraint, gives the system of equations to solve in positive numbers:

$$\begin{cases} 2xyz = \lambda \\ x^2z = \lambda \\ x^2y = \lambda \\ x + y + z = 100 \end{cases}$$

The second and third equations tell us (since x is said to be positive, we can divide by it with impunity) that $y = z$; the first and second together then tell us $x = 2y$, so our constraint becomes $2y + y + y = 100$, or $y = 25$; then $z = y = 25$ and $x = 2y = 50$.