1. **(20 points)** *Calculate the following integrals, using whatever approach you find most effective:*

   (a) **(3 points)** $\iint_D xydA$ *where D is the rectangle with corners $(-2,1)$, $(-2,4)$, $(1,4)$, and $(1,1)$.*
   Since this is an integral over a rectangular region given by $-2 \leq x \leq 1$ and $1 \leq y \leq 4$, the integral can be set up as a simple iterated integral with constant limits:

$$\begin{aligned}\iint_D xydA &= \int_{-2}^{1}\int_{1}^{4} xydydx \\ &= \int_{-2}^{1} \left.\frac{xy^2}{2}\right]_{y=1}^{y=4} dx = \int_{-2}^{1} \frac{15}{2}xdx \\ &= \left.\frac{15}{4}x^2\right]_{-2}^{1} dx = \frac{-45}{4}\end{aligned}$$

   (b) **(6 points)** $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} x^2 - y^2 dydx$.
   It is possible but extremely unpleasant to sole this iterated integral directly (trigonometric substitution would become necessary for the outermost integral). It is easier, and generally more satisfactory, to rephrase it as a polar integral: the region D given by the inequalities $-3 \leq x \leq 3$ and $0 \leq y \leq \sqrt{9-x^2}$ is fairly immediately recognizable as the upper half of a disk centered around the origin with radius 3. This same region can be described in polar coordinates as constrained by $0 \leq r \leq 3$ and $0 \leq \theta \leq \pi$. Thus:

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} x^2 - y^2 dydx = \iint_D x^2 - y^2 dA = \int_0^{\pi}\int_0^3 r[(r\cos\theta)^2 - (r\sin\theta)^2]drd\theta$$

   The integrand $r[(r\cos\theta)^2 - (r\sin\theta)^2]$ can be simplified to $r^3\cos 2\theta$ by algebraic simplification and the cosine double-angle formula. Thus:

$$\begin{aligned}\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} x^2 - y^2 dydx &= \int_0^{\pi}\int_0^3 r^3\cos 2\theta drd\theta \\ &= \int_0^{\pi} \frac{81}{4}\cos 2\theta d\theta \\ &= \int_0^{\pi} \frac{81}{8}\sin 2\theta d\theta \\ &= \frac{81}{8}(\sin 2\pi - \sin 0) = 0\end{aligned}$$

   (c) **(6 points)** $\iint_D x+ydA$ *where D is the region bounded by the curves $x = y^3$ and $x = 2y^2 - y$.*
   This is a type II integral, since the region's nonconstant bounds phrased with x written as a function of y. The constant y-boundaries of this region are those values of y where the two boundaries intersect; we thus need to determine when $y^3 = 2y^2 - y$, or when $y^3 - 2y^2 + y = 0$. This is easy enough to factor into $y(y-1)^2 = 0$, so these two functions intersect at $y = 0$ and $y = 1$. Thus, our region can be described by the inequalities

$0 \le y \le 1$ and $2y^2 - y \le x \le y^3$, to give the iterated integral:

$$\begin{aligned}\int_0^1 \int_{2y^2-y}^{y^3} x + y dx dy &= \int_0^1 \frac{x^2}{2} + xy\Big]_{2y^2-y}^{y^3} dy \\ &= \int_0^1 \frac{y^6 - (4y^4 - 4y^3 + y^2)}{2} + (y^3 - 2y^2 + y)y dy \\ &= \int_0^1 \frac{y^6}{2} - y^4 + \frac{y^2}{2} dy = \frac{1}{12} - \frac{1}{5} + \frac{1}{6} = \frac{1}{20}\end{aligned}$$

(d) **(6 points)** $\iiint_E \frac{zy}{x} dV$ *where E is the solid bounded by the surfaces $x = 0$, $y = x$, $z = x^2$, and $z = 4$.*

As phrased, these surfaces do not in fact bound a finite region: there needs to be another boundary on y. However, we could bound x by either $0 \le x \le 2$ or $-2 \le x \le 0$ (exactly which would depend on the y-boundary), and bound z by $x^2 \le z \le 4$, so except fr the mysterious missing boundaries on y, it would be easy to set up the integral:

$$\int_0^2 \int_?^? \int_{x^2}^4 \frac{zy}{x} dz dy dx$$

2. **(15 points)** *Write but do not evaluate the following integrals:*

(a) **(4 points)** *A cylindrical integral to calculate the volume of the solid which lies between the sheets of the hyperboloid $x^2 + y^2 - z^2 = -9$ and within the cylinder $x^2 + y^2 = 16$.*

The two sheets of this hyperboloid are given by $z = \pm\sqrt{x^2 + y^2 + 9} = \pm\sqrt{r^2 + 9}$, so the boundaries on z are $-\sqrt{r^2 + 9} \le z \le \sqrt{r^2 + 9}$. The boundaries on r and θ, however, are easy: since we lie inside the cylinder $r^2 = 16$, our constraints are $0 \le r \le 4$ and the usual radially-symmetric limits $0 \le \theta \le 2\pi$, so the integral is:

$$\int_0^{2\pi} \int_0^4 \int_{-\sqrt{r^2+9}}^{\sqrt{r^2+9}} r dz dr d\theta$$

This is not actually too difficult to evaluate by using a u-substitution for the secodn integral; the result is $\frac{392\pi}{3}$.

(b) **(4 points)** *A spherical integral to calculate $\iiint_E z dV$ where E is the section of the sphere $x^2 + y^2 + z^2 = 25$ where $0 \le x \le y$, and $z \ge 0$.*

Because x and y are non-negative, we are considering only first-quadrant values of θ, i.e. between 0 and $\frac{\pi}{2}$. Since we furthermore are saying $x \le y$, we are looking only at that part of the first quadrant above the midline $y = x$; this would be the sliver $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$. Since $z \ge 0$, we are also looking at values of ϕ between 0 (the north pole) and $\frac{\pi}{2}$ (the equator). Finally, our bounds on ρ are simple: since $0 \le x^2 + y^2 + z^2 \le 25$, $0 \le \rho \le 5$. Thus we have the integral:

$$\int_{\pi/4}^{\pi/2} \int_0^{\pi/2} \int_0^5 (\rho^2 \sin\phi)(\rho\cos\phi) d\rho d\phi d\theta$$

which could be evaluated by using the sine double-angle formula to get $\frac{625\pi}{32}$.

(c) **(4 points)** *A Cartesian (rectangular) form of the integral $\int_{\pi/2}^{\pi} \int_0^{\pi/6} \int_0^{4\sec\phi} \rho^3 \sin\phi\cos\phi d\rho d\phi d\theta$.*

The integrand is $(\rho\cos\phi)(\rho^2\sin\phi)$; as a Cartesian form, the integrand would lose the integrating factor $\rho^2\sin\phi$, leaving behind $\rho\cos\phi = z$; thus our integral is $\iiint_E z dV$ where E is described by the spherical constraints $0 \le \rho \le 4\sec\phi$, $0 \le \phi \le \frac{\pi}{6}$, and $\frac{\pi}{2} \le \theta \le \pi$. One of these constrains is easy to interpret: $\frac{\pi}{2} \le \theta \le \pi$ describes the second quadrant, so $y \ge 0$ and $x \le 0$. The second=easiest to interpret is $\rho \le 4\sec\phi$; multiplying both sides by $\cos\phi$, we see that this boundary is $\rho\cos\phi \le 4$, or $z \le 4$. Finally, z is bounded below by the ϕ-constraint: since $\phi \le \frac{\pi}{6}$, $\tan\phi \le \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, so $\tan^2\phi \le \frac{1}{3}$, and thus $\rho^2\sin^2\phi \le \frac{1}{3}\rho^2\cos^2\phi$, and thus $3(x^2+y^2) \le z^2$. So $3x^2+3y^2 \le z \le 4$; since these boundaries coincide at $x^2+y^2 = \frac{4}{3}$, we have boundaries on x and y of $-\sqrt{\frac{4}{3}} \le x \le 0$ and $0 \le y \le \sqrt{\frac{4}{3} - x^2}$ giving the integral:

$$\int_{-\sqrt{4/3}}^{0} \int_{0}^{\sqrt{4/3-x^2}} \int_{\sqrt{3x^2+3y^2}}^{4} z dz dy dx$$

(d) **(3 points)** *A polar integral to calculate $\int_E 2xy dA$, where E is the region given by $4 \le x^2+y^2 \le 16$ with $x \ge 0$ and $y \le 0$.*

This region is the fourth quadrant semi-circle with radius 4, so $0 \le r \le 4$ and $\frac{3\pi}{2} \le \theta \le 2\pi$, giving the integral:

$$\int_{3\pi/2}^{2\pi} \int_0^4 2(r\cos\theta)(r\sin\theta) r dr d\theta$$

Using the double-angle formula, this integrand could be rewritten as $r^3\sin 2\theta$. The integral could be evaluated to give 64.

3. **(5 points)** *Using the transformations $x = \frac{u^2}{v}$, $y = \frac{v}{u}$, evaluate $\iint_D y^2 dA$ over the region D bounded by $xy = 1$, $xy = 2$, $y = 1$, and $xy^2 = 2$.*

In the uv-plane, these boundaries are respectively $\frac{u^2}{v}\frac{v}{u} = 1$, $\frac{u^2}{v}\frac{v}{u} = 2$, $\frac{v}{u} = 1$, and $\frac{u^2}{v}\left(\frac{v}{u}\right)^2 = 2$. These can be algebraically simplified to $u = 1$, $u = 2$, $v = u$, and $v = 2$, so the region is described by $1 \le u \le 2$ and $u \le v \le 2$. Now we calculate the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2u}{v} & \frac{-u^2}{v^2} \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{2}{v} - \frac{1}{v} = \frac{1}{v}$$

so the integral can be rewritten

$$\begin{aligned} \iint_D y^2 dA &= \int_1^2 \int_u^2 \left(\frac{v}{u}\right)^2 \frac{1}{v} dv du \\ &= \int_1^2 \int_u^2 \frac{v}{u^2} dv du \\ &= \int_1^2 \frac{4-u^2}{2u^2} du \\ &= \int_1^2 \frac{2}{u^2} - \frac{1}{2} du \\ &= \frac{-2}{2} + \frac{2}{1} - \frac{1}{2} = \frac{1}{2} \end{aligned}$$

4. **(6 points)** *Determine whether each of the following vector fields as either conservative or nonconservative; for each that is conservative, find a potential function:*

- $F(x,y) = e^x \cos y\mathbf{i} - e^x \sin y\mathbf{j}$.
  We calculate $\frac{\partial}{\partial y} e^x \cos y = e^x \sin y$, and $\frac{\partial}{\partial x}(-e^x \sin y) = e^x \sin y$, so this vector field is conservative. Using the "partial integrals" of each component, we find that the potential function $f(x,y)$ is given by
  $$f(x,y) = \int e^x \cos y dx = e^x \cos y + C(y)$$
  and
  $$f(x,y) = \int -e^x \sin y dy = e^x \cos y + D(x)$$
  so $f(x,y) = e^x \cos y$ (or any constant added thereto, is the potential function of $F(x,y)$.
- $G(x,y) = \langle 3x^2 + 2y^2, 4xy + 3\rangle$.
  We calculate $\frac{\partial}{\partial y} 3x^2 + 2y^2 = 4y$, and $\frac{\partial}{\partial x} 4xy + 3 = 4y$, so this vector field is conservative. Using the "partial integrals" of each component, we find that the potential function $g(x,y)$ is given by
  $$g(x,y) = \int 3x^2 + 2y^2 dx = x^3 + 2xy^2 + C(y)$$
  and
  $$g(x,y) = \int 4xy + 3dy = 2xy^2 + 3y + D(x)$$
  Although these appear different, the term x^3 in the first integral is represented in the second within the junk term $D(x)$, and likewise the term $3y$ in the second integral is subsumed into the first integral's junk term $C(y)$. Thus, a potential function that matches both descriptions is $g(x,y) = x^3 + 2xy^2 + 3y$.
- $H(x,y) = \frac{x}{y}\mathbf{i} + \ln y\mathbf{j}$. We calculate $\frac{\partial}{\partial y}\frac{x}{y} = \frac{-x}{y^2}$, and $\frac{\partial}{\partial x} \ln y = 0$, so this vector field is nonconservative, since these results are nonequal.

5. **(14 points)** *Calculate the following path integrals.*

   (a) **(4 points)** $\int_C x^2 - y^2 dx$ *where C is the curve $y = x^2$ from $(1,1)$ to $(3,9)$.*
   A parameterization of this curve is $x = t$, $y = t^2$, for the interval $1 \leq t \leq 3$. Then this path integral is
   $$\begin{aligned}\int_C x^2 - y^2 dx &= \int_1^3 [x(t)^2 - y(t)^2]x'(t)dt \\ &= \int_1^3 (t^2 - t^4)(1)dt \\ &= \frac{t^3}{3} - \frac{t^5}{5}\bigg]_1^3 = \frac{26}{3} - \frac{242}{5}\end{aligned}$$

   (b) **(5 points)** $\int_C x^2 ds$ *where C is the line segment from $(1,2)$ to $(0,5)$.*

A simple parameterization of this curve is $x = 1 - t$ and $y = 2 + 3t$, with $0 \leq t \leq 1$. Then this integral is:

$$\begin{aligned}\int_C x^2 ds &= \int_0^1 x(t)^2 \sqrt{x'(t)^2 + y'(t)^2} dt \\ &= \int_0^1 (1-t)^2 \sqrt{(-1)^2 + 3^2} dt \\ &= \sqrt{10} \int_0^1 (1 - 2t + t^2) dt = \sqrt{10}(1 - 1 + \frac{1}{3}) = \frac{\sqrt{10}}{3}\end{aligned}$$

(c) **(5 points)** $\int_C F \cdot d\mathbf{r}$, *where* $F(x, y, z) = y\mathbf{i} + xz\mathbf{j} - 3\mathbf{k}$ *and* C *is the curve given by* $y = x^2$ *and* $z = x^3$ *between* $(0, 0, 0)$ *and* $(1, 1, 1)$.

This curve is parameterized as $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ for $0 \leq t \leq 1$, so the integral can be evaluated as

$$\begin{aligned}\int_C F \cdot d\mathbf{r} &= \int_0^1 F(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^1 (t^2)(1) + (tt^3)(2t) - 3(3t^2) dt = \int_0^1 2t^5 - 8t^2 \\ &= \frac{1}{3} - \frac{8}{3} = \frac{-7}{3}\end{aligned}$$