1. **(20 points)** *Consider the vector-valued function* $\mathbf{r}(t) = \left\langle \frac{t^3}{3}, -2t, t^2 \right\rangle$.

   (a) **(5 points)** *Find the parametric equations of a tangent line to the curve described by* $\mathbf{r}(t)$ *at the point* $(9, -6, 9)$.

   We are looking at the curve at $t = 3$: necessary information will be the point of tangency (which we were given), and the orientation of the tangent line, which is given by $\mathbf{r}'(3)$. Since $\mathbf{r}'(t) = \langle t^2, -2, 2t\rangle$, in this particular case our orientation is $\langle 9, -2, 6\rangle$, so our parametric equation is

$$\begin{cases} x = 9t + 9 \\ y = -2t - 6 \\ z = 6t + 9 \end{cases}$$

   (b) **(5 points)** *Find the arclength along this curve from* $(0, 0, 0)$ *to* $(9, -6, 9)$.

   Above we determined $|\mathbf{r}'|$; now we calculate $|\mathbf{r}'(t)|$:

$$|\mathbf{r}'(t)| = \sqrt{(t^2)^2 + (-2)^2 + (2t)^2} = \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2 + 2)^2} = t^2 + 2$$

   and thus our arclength is

$$\int_0^3 (t^2 + 2)dt = \frac{1}{3}t^3 + 2t\Big]_0^3 = (9 + 6) - (0 + 0) = 15$$

   (c) **(5 points)** *Find the curvature of this curve at* $(9, -6, 9)$.

   While there are other formulas for the curvature, the easiest,e specially given our knowledge so far, is

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}.$$

   Note that $\mathbf{r}'(t) = \langle t^2, -2, 2t\rangle$ and $|\mathbf{r}'(t)| = t^2 + 2$ were calculated above; $\mathbf{r}''(t) = \langle 2t, 0, 2\rangle$ follows readily. Evaluating each of these at $t = 3$, we can find the curvature at the point $t = 3$ with ease:

$$\kappa = \frac{|\langle 9, -2, 6\rangle \times \langle 6, 0, 2\rangle|}{3^3} = \frac{|\langle -4, 18, 12\rangle|}{(3^2 + 2)^3} = \frac{2\sqrt{16 + 81 + 36}}{11^3} = \frac{2}{121}$$

   (d) **(5 points)** *Find the unit binormal vector at* $(9, -6, 9)$.

   Since we calculated $\mathbf{r}' \times \mathbf{r}''$ above, we already know a vector with the right orientation: $\langle 4, 2, 4\rangle$. Making it of unit length is a matter of calculating

$$\mathbf{B} = \frac{\langle -4, 18, 12\rangle}{|\langle -4, 18, 12\rangle|} = \left\langle \frac{-2}{11}, \frac{9}{11}, \frac{6}{11} \right\rangle$$

2. **(10 points)** *Consider the line given by the system of parametric equations* $x = 3t - 5$, $y = -t + 2$, $z = 4t + 4$, *and the plane given by the equation* $2x + 2y - z = 8$.

   (a) **(3 points)** *Does the line intersect the plane or not?*

We substitute the parametric equations for x, y, and z given by the line into the equation of the plane, and see if there is a t satisfying the equation:

$$\begin{aligned} 2(3t-5)+2(-t+2)-(4t+4) &= 8 \\ 6t-10-2t+4-4t-4 &= 8 \\ -10 &= 8 \end{aligned}$$

Since there is no value of t making this equation true (or if there is, I'd like to know about it!), there is no value of t such that the point $(3t-5, -t+2, 4t+4)$ lies in the plane $2x+2y-z=8$, so the line does not intersect the plane.

(b) **(7 points)** *If the line intersects the plane, determine the point where it does so; if it does not intersect the plane, determine the distance between the line and plane.*

Let us pick an arbitrary point on the line; we might use the $t=0$ point $(-5, 2, 4)$ for simplicity. Likewise pick an arbitrary point on the plane; there are several obvious choices, but in this particular solution, let's consider $(4, 0, 0)$. The vector $\mathbf{u}$ between these points is $\langle -9, 2, 4\rangle$. This vector includes significant components parallel to the line and plane, so it doesn't reflect the *shortest* distance between the line and plane; to find that, we need to find the length of the projection of $\mathbf{u}$ onto the line and plane's mutual normal. The plane's normal is clearly $\mathbf{n} = \langle 2, 2, -1\rangle$; a cursory inspection indicates that it is perpendicular to the line as well (an aside: if the plane's normal were *not* perpendicular to the line, the line would necessarily intersect the plane, so we can take the fact that $\mathbf{n}$ is orthogonal to the line as given). Now we simply calculate the length of the normal component of $\mathbf{u}$:

$$\text{comp}_{\mathbf{n}}\,\mathbf{u} = \frac{|\mathbf{n}\cdot\mathbf{u}|}{|\mathbf{n}|} = \frac{|2(-9)+2\cdot 2-1\cdot 4|}{\sqrt{2^2+2^2+(-1)^2}} = \frac{18}{3} = 6$$

3. **(15 points)** *In the questions which follow,* $\mathbf{u} = 2\mathbf{i}-4\mathbf{j}+6\mathbf{k}$ *and* $\mathbf{v} = \mathbf{i}-\mathbf{k}$.

(a) **(4 points)** *Identify each of the following four expressions as a vector, a scalar, or as uncalculatable nonsense. You do not need to calculate these expressions or justify your assertions!*

i. $(\mathbf{u}\cdot\mathbf{v}) - |\mathbf{v}|$.
Working from the inside of each expression out: $\mathbf{u}\cdot\mathbf{v}$ is a scalar (dot product of two vectors), $|\mathbf{v}|$ is a scalar (length of a vector), and the difference of these two is a scalar minus a scalar (ordinary subtraction), which yields a scalar.

ii. $\frac{\mathbf{u}-\mathbf{v}}{\mathbf{v}}$.
Eyeballing this, we immediately see division by the vector $\mathbf{v}$, which is a nonsensical operation (there is no idea of what it means to divide by a vector).

iii. $|\mathbf{u}|\mathbf{v} - (\mathbf{u}\times\mathbf{v})$.
Working from the inside of each expression out: $|\mathbf{u}|$ is a scalar (length of a vector), so $|\mathbf{u}|\mathbf{v}$ is the product of a scalar and a vector, which is a vector (scalar-product of a vector). $\mathbf{u}\times\mathbf{v}$ is a vector (cross product of a vector), so the difference of the vectors $|\mathbf{u}|\mathbf{v}$ and $\mathbf{u}\times\mathbf{v}$ (vector subtraction) is a vector.

iv. $\frac{1}{|\mathbf{u}|}(\mathbf{u}\times\mathbf{v})$.
Working from the inside of each expression out: $|\mathbf{u}|$ is a scalar (length of a vector), and $\frac{1}{|\mathbf{u}|}$ is its reciprocal, which is also a scalar. $\mathbf{u}\times\mathbf{v}$ is a vector (cross product of

a vector), so the product of a scalar and a vector is the scalar-product of a vector, which is a vector.

(b) **(3 points)** *Calculate* $\mathbf{u} \times \mathbf{v}$.

Using any method you wish, one can find this to be $4\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$.

(c) **(4 points)** *Find the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

The angle is given by $\theta = \cos^{-1} \frac{\mathbf{u}\cdot\mathbf{v}}{|u||v|}$, which we can calculate as a matter of routine:

$$\theta = \cos^{-1} \frac{2 \cdot 1 - 4 \cdot 0 + 6(-1)}{\sqrt{2^2 + 4^2 + 6^2}\sqrt{1^2 + 0^2 + 1^2}} = \cos^{-1} \frac{-4}{\sqrt{56}\sqrt{2}} = \cos^{-1} \frac{-1}{\sqrt{7}}$$

(d) **(4 points)** *Calculate* $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$.

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \frac{2 \cdot 1 - 4 \cdot 0 + 6(-1)}{1 \cdot 1 + 0 \cdot 0 + (-1)(-1)}\langle 1, 0, -1\rangle = \langle -2, 0, 2\rangle$$

4. **(20 points)** *Answer the following questions:*

(a) **(5 points)** *Evaluate the limit* $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2}$ *or demonstrate that it does not exist.*

We can approach the point $(0, 0)$ along any of the lines $y = mx$ and take the limit of the above expression as $x \to 0$ to find:

$$\lim_{x\to 0} \frac{2x^2(mx)}{x^2 + (mx)^2} = \lim_{x\to 0} \frac{2mx^3}{(1 + m^2)x^2} = \lim_{x\to 0} \frac{2mx}{1 + m^2} = \frac{2m}{1 + m^2} \cdot 0 = 0$$

Similarly, we can look at the approach along the vertical line $x = 0$ as $y \to 0$:

$$\lim_{y\to 0} \frac{2 \cdot 0^2 y}{0^2 + y^2} = \lim_{y\to 0} \frac{0y}{y^2} = \lim_{y\to 0} \frac{0}{y} = 0$$

Since every approach yields zero, we know this limit is zero. More formally, we could show that when $\sqrt{x^2 + y^2} < \delta$, since x and y themselves must lie between $-\delta$ and δ, it follows that $\left|\frac{2x^2y}{x^2+y^2}\right| < \frac{2\delta^3}{\delta^2} = 2\delta$, so we can guarantee $\frac{2x^2y}{x^2+y^2}$ to be within any chosen ϵ of 0 by looking at (x, y) within a distance of $\frac{\epsilon}{2}$ of $(0, 0)$.

(b) **(5 points)** *Given the trajectory* $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 3t\mathbf{k}$, *identify the tangential and normal components of the acceleration vector when* $t = 2$.

Note $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3\mathbf{k}$, and $\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{j}$. In particular, $\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{a}(2) = 2\mathbf{j}$.

Since $\mathbf{r}'(2)$ is a (non-unit-length) tangent to the trajectory, we can find the tangential component of $\mathbf{a}(2)$ by means of a projection:

$$\mathbf{a}_T(2) = \operatorname{proj}_{\mathbf{r}'(2)} \mathbf{a}(2) = \frac{\mathbf{r}'(2) \cdot \mathbf{a}(2)}{\mathbf{r}'(2) \cdot \mathbf{r}'(2)}\mathbf{r}'(2) = \frac{1 \cdot 0 + 4 \cdot 2 + 3 \cdot 0}{1 \cdot 1 + 4 \cdot 4 + 3 \cdot 3}(\mathbf{i}+4\mathbf{j}+3\mathbf{k}) = \frac{4}{13}\mathbf{i}+\frac{16}{13}\mathbf{j}+\frac{12}{13}\mathbf{k}$$

and since we have identified the tangential component, everything that remains is normal:

$$\mathbf{a}_N(2) = \mathbf{a}(2) - \mathbf{a}_T(2) = -\frac{4}{13}\mathbf{i} + \frac{10}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$$

If you wish, it is possible to verify that this component is normal by calculating $\mathbf{a}_N(2)\cdot\mathbf{r}'(2)$ and noting that it is zero.

(c) **(5 points)** *Find the equation of the tangent plane to the curve* $z = 2x^2 - 3xy$ *at* $(2, -1, 14)$.
Letting $f(x, y) = 2x^2 - 3xy$, we know the tangent plane has the equation

$$z - f(2, -1) = f_x(2, -1)(x - 2) + f_y(2, -1)(y - (-1))$$

and we've already been given $f(2, -1) = 14$ as a bonus. Easy partial calculations tell us $f_x(x, y) = 4x - 3y$ and $f_y(x, y) = -3x$, so in particular $f_x(2, -1) = 11$ and $f_y(2, -1) = -6$, which substituted back into our original equation gives

$$z - 14 = 11(x - 2) - 6(y + 1)$$

This can, but need not, be reformulated as $z = 11x - 6y - 14$.

(d) **(5 points)** *Find an equation of the tangent plane to the surface* $x^2 + 3y^2 + xz - z^2 = 12$ *at the point* $(-3, 1, 0)$.
Let $f(x, y, z) = x^2 + 3y^2 + xz - z^2$; then the above surface is the level surface $f(x, y, z) = 12$. Tangent planes to a level surface have the gradient at the point of tangency as their normal vectors; thus the normal vector to this plane will be given by $\nabla f(-3, 1, 0)$. Since $\nabla f(x, y, z) = \langle 2x + z, 6y, x - 2z\rangle$, it is easy to determine that the normal vector we seek is $\langle -6, 6, -3\rangle$. Since we know the plane's normal vector and we know it passes through the point $(-3, 1, 0)$, we can get an equation of the plane with ease:

$$-6(x + 3) + 6(y - 1) - 3z = 0$$

5. **(15 points)** *Answer the following questions about optimization:*

(a) **(7 points)** *Find the critical points of* $g(x, y) = x^2 - 3xy + y^3$ *and identify each as a local maximum, local minimum, or saddle point.*
Note that $\nabla g(x, y) = \langle 2x - 3y, 3y^2 - 3x\rangle$. The critical points are those pairs (x, y) where $\nabla g(x, y) = \mathbf{0}$; that is, where both of the equations:

$$\begin{cases} 2x - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases}$$

From the second equation, it is clearly necessary that $x = y^2$; substituting this into the first gives the quadratic $2y^2 - 3y = 0$, which has solutions $y = 0$ and $y = \frac{3}{2}$, which have corresponding x-values of $0^2 = 0$ and $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$; thus the critical points are $(0, 0)$ and $(\frac{9}{4}, \frac{3}{2})$.
Now we need to consider the criterion function $D = g_{xx}(x, y)g_{yy}(x, y) - [g_{xy}(x, y)]^2$ at each of these points. Fortunately, it is easy to calculate: $g_{xx} = 2$, $g_{yy} = 6y$, and $g_{xy} = -3$, so $D = 12y - 9$. Testing at $(0, 0)$, we get $D = -9 < 0$, so $(0, 0)$ will be a saddle point. Testing at $(\frac{9}{4}, \frac{3}{2})$, we get $D = 9 > 0$, so $(\frac{9}{4}, \frac{3}{2})$ is some manner of extremum; using the traditional second-derivative test, we know $g_{xx} > 0$, so it is specifically a local minimum.

(b) **(8 points)** *Find positive* x, y, *and* z *such that* $xyz = 1$ *and which minimize the value of* $12x + 9y + 2z$.
This is a constrained optimization problem; the constraint if $xyz = 1$, so we will define $g(x, y, z) = xyz$; the goal function is $f(x, y, z) = 12x + 9y + 2z$. To solve this we will use Lagrange multipliers:

$$\nabla f = \lambda \nabla g$$

which, for these specific values of f and g, gives

$$\langle 12, 9, 2\rangle = \lambda\langle yz, xz, xy\rangle$$

which, together with our constraint, gives the system of equations to solve in positive numbers:

$$\begin{cases} 12 = \lambda yz \\ 9 = \lambda xz \\ 2 = \lambda xy \\ xyz = 1 \end{cases}$$

Note that none of these variables can be zero, so division is safe. Dividing any two of the first three equations will give simple relationships among x, y, and z: for instance, the quotient of the first two will reveal that $y = \frac{12}{9}x$; likewise, the quotient of the first and third reveals that $z = 6x$. Then, plugging these into the last equation, we see that $x\left(\frac{12}{9}x\right)(6x) = 1$, so $8x^3 = 1$, and thus $x = \frac{1}{2}$, $y = \frac{2}{3}$, and $z = 3$.

6. **(10 points)** *Calculate the following integrals:*

(a) **(4 points)** *Calculate* $\iint_A (xy^2 - \frac{1}{x})dA$ *if* A *is the rectangle* $1 \le x \le 3, -2 \le y \le 1$.
Using iterated integrals:

$$\begin{aligned} \iint_A (xy^2 - \frac{1}{x})dA &= \int_1^3 \int_{-2}^1 xy^2 - \frac{1}{x}dydx \\ &= \int_1^3 \frac{xy^3}{3} - \frac{y}{x}\Big]_{y=-2}^{y=1} dx \\ &= \int_1^3 3x - \frac{3}{x}dx \\ &= \frac{3}{2}x^2 - 3\ln x\Big]_1^3 = 12 - 3\ln 3 \end{aligned}$$

(b) **(6 points)** *Find the integral of* xy *over the region bounded by the curves* $y = x$ *and* $y = x^4$.
This region lies between $(0, 0)$ and $(1, 1)$, as can be determined by finding the solutions of $x = x^4$. It is presented as a Type I integral, which is a sound way of approaching it: the "hard" bounds on the left and right are $x = 0$ and $x = 1$, while the nonconstant bounds are $y = x$ above and $y = x^4$ below, so we could set up the integral as such:

$$\begin{aligned} \iint_D xydA &= \int_0^1 \int_x^{x^4} xydydx \\ &= \int_0^1 x\frac{y^2}{2}\Big]_{y=x^4}^{y=x} dx \\ &= \int_0^1 x\frac{x^2}{2} - x\frac{(x^4)^2}{2}dx \\ &= \int_0^1 \frac{x^3}{2} - \frac{x^9}{2}dx \\ &= \frac{x^4}{8} - \frac{x^{10}}{20}\Big]_0^1 = \left(\frac{1}{8} - \frac{1}{20}\right) - (0 - 0) = \frac{3}{40} \end{aligned}$$

Note that the above integral could also be set up as the Type II form $\int_0^1 \int_{\sqrt[4]{y}}^{y} xy dx dy$, but is much more difficult to evaluate that way.

7. **(20 points)** *Set up but do not evaluate the following integrals.*

(a) **(4 points)** *An integral to determine the volume of the solid enclosed by the planes $y = 2x$, $z = 0$, $x = 0$, and $z = 4 - y$.*

Looking at the plane which forms the lower bound, $z = 0$, we see that the other three planes intersect that plane in the lines $y = 2x$, $x = 0$, and $y = 4$. Our region in the plane is thus the triangle with coordinates $(0,0)$, $(0,4)$, and $(2,4)$; the only nonconstant bound is the bound on the bottom (if performing a Type I integral) or on the right (if performing a Type II integral) by the line $y = 2x$. It is most natural to set it up as a Type I integral, in which case the left and right bounds are $x = 0$ and $x = 2$, the upper bound is $y = 4$, and the lower bound is $y = 2x$. Since the height of the solid at a point (x, y) is given by the difference between $4 - y$ and 0, the integrand is $4 - y$, yielding:

$$\int_0^2 \int_{2x}^4 4 - y dy dx$$

Alternatively, we could set up a Type II integral; then the bounds on y are 0 and 4, while the bounds on x are 0 and $\frac{y}{2}$, giving:

$$\int_0^4 \int_0^{y/2} 4 - y dx dy$$

If one *really* wanted to be perverse, even polar coordinates are an option: the line $y = 2x$ corresponds with the boundary $\theta = \arctan 2$, while $x = 0$ corresponds with $\theta = \frac{\pi}{2}$; the bound $y = 4$ is $r \sin\theta = 4$, or $r = 4\csc\theta$, so we could set up this integral as

$$\int_{\arctan 2}^{\pi/2} \int_0^{4\csc\theta} r(4 - r\sin\theta) dr d\theta$$

(b) **(5 points)** *A polar iterated form of $\iint_R x^2 - y^2 dA$, where A is the region where $x \geq 0$, $y \geq 0$, and $1 \leq x^2 + y^2 \leq 16$.*

The condition $1 \leq x^2 + y^2 \leq 16$ describes an annulus (or "ring") with inner radius 1 and outer radius 16. The additional conditions $x \geq 0$, $y \geq 0$ means that we're looking at the quarter-section of this annulus lying in the first quadrant. Fortunately, this shape lends itself easily to polar boundaries; the first quadrant corresponds to $0 \leq \theta \leq \frac{\pi}{2}$, while the annulus in question is given by $1 \leq r \leq 4$, so together with the substitutions $x = r\cos\theta$ and $y = r\sin\theta$, we have an easy construction for our integral:

$$\int_0^{\pi/2} \int_1^4 r[(r\cos\theta)^2 - (r\sin\theta)^2] dr d\theta$$

The order of the integrations can be switched here, if desired; also, the integrand can be simplified to $r^3(\cos^2\theta - \sin^2\theta)$, or even to $r^3\cos(2\theta)$.

(c) **(5 points)** *Set up (but do not evaluate) a cylindrical form for the integral $\iiint_E (x+1) dV$ over the solid which lies below the paraboloid $z = 8 - x^2 - y^2$ and above the paraboloid $z = x^2 + y^2$.*

These paraboloids intersect when $8 - x^2 - y^2 = x^2 + y^2$; that is, when $x^2 + y^2 = 4$, or in the parlance of polar coordinates, when $r^2 = 4$. So our range of r-values relevant for describing this solid is $[0, 2]$. The solid is radially symmetric, since both of the paraboloids described are radially symmetric, and thus the standard full interval $0 \leq \theta \leq 2\pi$ suffices. The lower surface is described by $z = r^2$, and the upper by $z = 8 - r^2$, so the bounds on z are $r^2 \leq z \leq 8 - r^2$, and so we can now express the integral in cylindrical notation:

$$\iiint_E (x+1)dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} (r\cos\theta + 1) r dz dr d\theta$$

This evaluates, incidentally, to 16π.

(d) **(5 points)** *Set up (but do not evaluate) a spherical integral to calculate the volume lying between the cones $z^2 = x^2 + y^2$ and $z^2 = 3x^2 + 3y^2$ for $z \geq 0$, and bounded above by the sphere $x^2 + y^2 + z^2 = 16$.*

The sphere $x^2 + y^2 + z^2 = 16$ can be converted, of course, to $\rho^2 = 16$ or $\rho = 4$. The cone $z^2 = x^2 + y^2$ translates to $\rho^2 \cos^2\phi = \rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta$, which can be converted via simple algebra and trigonometry to $\tan^2\phi = 1$, or $\tan\phi = \pm 1$; since we are only looknig at cases with $z > 0$, the negative tangent possibilities can be discarded, and the half-cone that remains corresponds to $\phi = \arctan 1 = \frac{\pi}{4}$.

Using an almost identical line of argument, the cone $z^2 = 3x^2 + 3y^2$ has spherical representation $\tan^2\phi = \frac{1}{3}$, so $\phi = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$. Thus, the solid between these two cones is described by the inequality $\frac{\pi}{6} \leq \phi \leq \frac{\pi}{4}$. We thus have boundaries on both ρ and ϕ, and, since this solid is radially symmetric, θ can take on the usual unrestricted limits to give the volume integral:

$$\iiint_E dV = \int_0^{2\pi} \int_{\pi/6}^{\pi/4} \int_0^4 \rho^2 \sin\phi d\rho d\phi d\theta$$

This integral, incidentally, is $\frac{64\pi(\sqrt{3}-\sqrt{2})}{3}$.

8. **(10 points)** *Calculate the following path integrals:*

(a) **(5 points)** $\int_C x^2 ds$ *where C is the line segment from $(0, 4)$ to $(3, 2)$.*

Let us parameterize this segment with $x = 3t$, $y = 4 - 2t$, for $0 \leq t \leq 1$. Then this integral is:

$$\begin{aligned}\int_C x^2 ds &= \int_0^1 x(t)^2 \sqrt{x'(t)^2 + y'(t)^2} dt \\ &= \int_0^1 9t^2 \sqrt{3^2 + (-2)^2} dt \\ &= 3t^3\sqrt{13}\Big]_0^1 = 3\sqrt{13}\end{aligned}$$

(b) **(5 points)** $\int_C F \cdot d\mathbf{r}$, *where $F(x, y, z) = \langle 4y + z, 3x - z, 2z \rangle$ and C is the curve given by $x = t$, $y = t^2$, and $z = t$ from $(0, 0, 0)$ to $(2, 4, 2)$.*

Since a parameterization is already given (and the curve in question is from $t = 0$ to $t = 2$, we may evaluate this directly:

$$\begin{aligned}\int_C F\cdot d\mathbf{r} &= \int_0^2 \langle 4y(t)+z(t), 3x(t)-z(t), 2z(t)\rangle\cdot\langle x'(t), y'(t), z'(t)\rangle dt\\ &= \int_0^2 \langle 4t^2+t, 2t, 2t\rangle\cdot\langle 1, 2t, 1\rangle dt\\ &= \int_0^2 8t^2+3tdt\\ &= \frac{8}{3}t^3+\frac{3}{2}t^2\Big]_0^2 = \frac{64}{3}+6=\frac{82}{3}\end{aligned}$$

9. **(15 points)** *Answer the following vector-field-related questions:*

(a) **(4 points)** *Determine whether the vector field* $G(x,y)=\langle \frac{2x}{y}+3, y^2-\frac{x^2}{y^2}\rangle$ *is conservative; if conservative, find a potential function.*

We calculate $\frac{\partial}{\partial y}(\frac{2x}{y}+3)=\frac{-2x}{y^2}$, and $\frac{\partial}{\partial x}(y^2-\frac{x^2}{y^2})=\frac{-2x}{y^2}$, so this vector field is conservative. Using the "partial integrals" of each component, we find that the potential function $g(x,y)$ is given by

$$g(x,y)=\int \frac{2x}{y}+3dx=\frac{x^2}{y}+3x+C(y)$$

and

$$g(x,y)=\int y^2-\frac{x^2}{y^2}dy=\frac{y^3}{3}+\frac{x^2}{y}+D(x)$$

Although these appear different, the term $3x$ in the first integral is represented in the second within the junk term $D(x)$, and likewise the term $\frac{y^3}{3}$ in the second integral is subsumed into the first integral's junk term $C(y)$. Thus, a potential function that matches both descriptions is $g(x,y)=\frac{x^2}{y}+3x+\frac{y^3}{3}$.

(b) **(7 points)** *Use Green's Theorem to evaluate* $\int_C x^2y^2dx+8xy^3dy$*, where* C *is the triangular path consisting of linear subpaths from* $(0,0)$ *to* $(1,3)$ *to* $(0,3)$ *and back to* $(0,0)$*.*

The described path is a counterclockwise path around the triangular region bounded by $(0,0)$, $(0,3)$, and $(1,3)$, so by Green's Theorem, if the above region is called D:

$$\int_C x^2y^2dx+8xy^3dy=\iint_D \frac{\partial}{\partial x}(8xy^3)-\partial\partial y(x^2y^2)dA=\iint_D 8y^3-2x^2ydA$$

This area integral can be iterated fairly easily: its left and right bounds are clearly $x=0$ and $x=1$, and it is bounded above by the line $y=3$, but it is bounded below by the line from $(0,0)$ to $(1,3)$; that is, $y=3x$. Thus, our limits can be phrased as such:

$$\begin{aligned}\iint_D 8y^3-2x^2ydA &= \int_0^1\int_{3x}^3 8y^3-2x^2ydydx\\ &= \int_0^1 (162-9x^2)-(162x^4-9x^4)dx\\ &= 162x-3x^3-\frac{153}{5}x^5\Big]_0^1 = 159-\frac{153}{5}\end{aligned}$$

(c) **(4 points)** *Find the divergence and curl of the vector field* $\mathbf{F}(x,y,z) = 0i + ye^x\mathbf{j} + ye^z\mathbf{k}$.

The curl is calculated as such:

$$\nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & ye^x & ye^z \end{vmatrix} // = \left(\frac{\partial}{\partial y}ye^z - \frac{\partial}{\partial z}ye^x\right)\mathbf{i} + \left(\frac{\partial}{\partial z}0 - \frac{\partial}{\partial x}ye^z\right)\mathbf{j} + \left(\frac{\partial}{\partial x}ye^x - \frac{\partial}{\partial y}0\right)\mathbf{k}$$
$$= e^z\mathbf{i} + ye^x\mathbf{k}$$

And the divergence is somewhat easier:

$$\nabla \cdot F = \frac{\partial}{\partial x}0 + \frac{\partial}{\partial y}ye^x + \frac{\partial}{\partial z}ye^z = e^x + ye^z$$