1. **(5 points)** *Evaluate the limit* $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2}$ *or demonstrate that it does not exist.*

   We can approach the point $(0,0)$ along any of the lines $y = mx$ and take the limit of the above expression as $x \to 0$ to find:

$$\lim_{x\to 0} \frac{2x^2(mx)}{x^2+(mx)^2} = \lim_{x\to 0} \frac{2mx^3}{(1+m^2)x^2} = \lim_{x\to 0} \frac{2mx}{1+m^2} = \frac{2m}{1+m^2} \cdot 0 = 0$$

   Similarly, we can look at the approach along the vertical line $x = 0$ as $y \to 0$:

$$\lim_{y\to 0} \frac{2\cdot 0^2 y}{0^2+y^2} = \lim_{y\to 0} \frac{0y}{y^2} = \lim_{y\to 0} \frac{0}{y} = 0$$

   Since every approach yields zero, we know this limit is zero. More formally, we could show that when $\sqrt{x^2+y^2} < \delta$, since x and y themselves must lie between $-\delta$ and δ, it follows that $\left|\frac{2x^2y}{x^2+y^2}\right| < \frac{2\delta^3}{\delta^2} = 2\delta$, so we can guarantee $\frac{2x^2y}{x^2+y^2}$ to be within any chosen ϵ of 0 by looking at (x,y) within a distance of $\frac{\epsilon}{2}$ of $(0,0)$.

2. **(5 points)** *Given the trajectory* $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 3t\mathbf{k}$*, identify the tangential and normal components of the acceleration vector when* $t = 2$.

   Note $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3\mathbf{k}$, and $\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{j}$. In particular, $\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{a}(2) = 2\mathbf{j}$.

   Since $\mathbf{r}'(2)$ is a (non-unit-length) tangent to the trajectory, we can find the tangential component of $\mathbf{a}(2)$ by means of a projection:

$$\mathbf{a}_T(2) = \operatorname{proj}_{\mathbf{r}'(2)} \mathbf{a}(2) = \frac{\mathbf{r}'(2)\cdot\mathbf{a}(2)}{\mathbf{r}'(2)\cdot\mathbf{r}'(2)}\mathbf{r}'(2) = \frac{1\cdot 0 + 4\cdot 2 + 3\cdot 0}{1\cdot 1 + 4\cdot 4 + 3\cdot 3}(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) = \frac{4}{13}\mathbf{i} + \frac{16}{13}\mathbf{j} + \frac{12}{13}\mathbf{k}$$

   and since we have identified the tangential component, everything that remains is normal:

$$\mathbf{a}_N(2) = \mathbf{a}(2) - \mathbf{a}_T(2) = -\frac{4}{13}\mathbf{i} + \frac{10}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$$

   If you wish, it is possible to verify that this component is normal by calculating $\mathbf{a}_N(2)\cdot\mathbf{r}'(2)$ and noting that it is zero.

3. **(5 points)** *Let* $u = xy\cos(yz) - xz^3$. *Calculate* $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, *and* $\frac{\partial u}{\partial z}$.

   x appears entirely in a context where it is being multiplied by other expressions, so u_x uses no fancy tools, jsut the observation that all expressions are linear in x:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(y\cos(yz)x) - \frac{\partial}{\partial x}(z^3x) = y\cos(yz) - z^3$$

   y appears nowhere in the subtrahend xz^3, so in the partial that "constant" term will vanish outright, but it appears in a complicated context in the minuend $xy\cos(yz)$, which will require the product rule and (implicitly) the chain rule:

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}(xy\cos(yz)) - \frac{\partial}{\partial y}(xz^3)\\ &= \left(\frac{\partial}{\partial y}(xy)\right)\cos(yz) + xy\frac{\partial}{\partial y}(\cos(yz)) - 0\\ &= x\cos(yz) + xy(-z\sin(yz)) - 0 = x\cos(yz) - xyz\sin(yz)\end{aligned}$$

z appears in each term of the subtraction, but neither of them are too complicated; we will use the chain rule implicitly and the power rule:

$$\frac{\partial u}{\partial z} = xy\left(\frac{\partial}{\partial z}\cos(yz)\right) - x\frac{\partial}{\partial z}z^3 = xy(-y\sin(yz)) - x(3z^2) = -xy^2\sin(yz) - 3xz^2$$

4. **(5 points)** *Find the equation of the tangent plane to the curve $z = 2x^2 - 3xy$ at $(2, -1, 14)$.*

Letting $f(x, y) = 2x^2 - 3xy$, we know the tangent plane has the equation

$$z - f(2, -1) = f_x(2, -1)(x - 2) + f_y(2, -1)(y - (-1))$$

and we've already been given $f(2, -1) = 14$ as a bonus. Easy partial calculations tell us $f_x(x, y) = 4x - 3y$ and $f_y(x, y) = -3x$, so in particular $f_x(2, -1) = 11$ and $f_y(2, -1) = -6$, which substituted back into our original equation gives

$$z - 14 = 11(x - 2) - 6(y + 1)$$

This can, but need not, be reformulated as $z = 11x - 6y - 14$.

5. **(2 point bonus)** *For functions $f(x, y)$ and $g(x, y)$ and a point (a, b) such that $f(a, b) = g(a, b) = 0$ but $g_x(a, b) \neq 0$ and $g_y(a, b) \neq 0$, prove that $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)}$ exists only if $f_x(a, b)g_y(a, b) = f_y(a, b)g_x(a, b)$.*

In order for $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)}$ to exist, it is necessary (if not sufficient) that in particular the approach along the line $y = b$, given by $\lim_{x\to a} \frac{f(x,b)}{g(x,b)}$, and the approach along the line $x = a$, given by $\lim_{y\to b} \frac{f(a,y)}{g(a,y)}$, be equal.

Each of these is a conventional single-variable limit and can be solved by conventional means. In particular, the evaluation of $\frac{f(x,b)}{g(x,b)}$ at $x = a$ gives the indeterminate form $\frac{0}{0}$, so we are justified in using L'Hôpital's Rule to find that

$$\lim_{x\to a} \frac{f(x, b)}{g(x, b)} = \lim_{x\to a} \frac{\frac{d}{dx}f(x, b)}{\frac{d}{dx}g(x, b)} = \lim_{x\to a} \frac{f_x(x, b)}{g_x(x, b)} = \frac{f_x(a, b)}{g_x(a, b)}$$

Note the final evaluation: since $g_x(a, b) \neq 0$ and thus the limit preceding it is *not* an indeterminate form and is subject to direct evaluation.

Using an identical method, we can find that $\lim_{y\to b} \frac{f(a,y)}{g(a,y)} = \frac{f_y(a,b)}{g_y(a,b)}$. Since we asserted above the two-dimensional limit only existed if these two limits both existed and were equal, we find that a necessary condition for the existence of the two-dimensional limit's existence is $\frac{f_x(a,b)}{g_x(a,b)} = \frac{f_y(a,b)}{g_y(a,b)}$; cross-multiplicaiton gives the desired equality.