1. **(4 points)** *Given $f(x,y) = 4x^2 + 2y^3 - 2xy - 1$, find the rate of change of the function at the point $(2,1)$ in the direction $\langle 1, -1\rangle$.*

   Using partial derivatives, one can calculate the gradient of f to be $\nabla f(x,y) = \langle 8x-2y, 6y^2-2x\rangle$. Evaluated specifically at the given point, we find that $\nabla f(2,1) = \langle 8\cdot 2 - 2\cdot 1, 6\cdot 1^2 - 2\cdot 2\rangle = \langle 14, 2\rangle$. Finally, we can calculate the directional derivative in a particular direction $\mathbf{u}$ by taking the dot product of the gradient with a unit-length vector in the correct direction: $\nabla f(2,1) \cdot \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{14\cdot 1 + 2\cdot(-1)}{\sqrt{1^2+(-1)^2}} = \frac{12}{\sqrt{2}}$.

2. **(5 points)** *Find an equation of the tangent plane to the surface $x^2 + 3y^2 + xz - z^2 = 12$ at the point $(-3, 1, 0)$.*

   Let $f(x,y,z) = x^2 + 3y^2 + xz - z^2$; then the above surface is the level surface $f(x,y,z) = 12$. Tangent planes to a level surface have the gradient at the point of tangency as their normal vectors; thus the normal vector to this plane will be given by $\nabla f(-3,1,0)$. Since $\nabla f(x,y,z) = \langle 2x+z, 6y, x-2z\rangle$, it is easy to determine that the normal vector we seek is $\langle -6, 6, -3\rangle$. Since we know the plane's normal vector and we know it passes through the point $(-3,1,0)$, we can get an equation of the plane with ease:

$$-6(x+3) + 6(y-1) - 3z = 0$$

3. **(6 points)** *Find the critical points of $g(x,y) = x^2 - 3xy + y^3$ and identify each as a local maximum, local minimum, or saddle point.*

   Note that $\nabla g(x,y) = \langle 2x - 3y, 3y^2 - 3x\rangle$. The critical points are those pairs (x,y) where $\nabla g(x,y) = \mathbf{0}$; that is, where both of the equations:

$$\begin{cases} 2x - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases}$$

   From the second equation, it is clearly necessary that $x = y^2$; substituting this into the first gives the quadratic $2y^2 - 3y = 0$, which has solutions $y = 0$ and $y = \frac{3}{2}$, which have corresponding x-values of $0^2 = 0$ and $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$; thus the critical points are $(0,0)$ and $(\frac{9}{4}, \frac{3}{2})$.

   Now we need to consider the criterion function $D = g_{xx}(x,y)g_{yy}(x,y) - [g_{xy}(x,y)]^2$ at each of these points. Fortunately, it is easy to calculate: $g_{xx} = 2$, $g_{yy} = 6y$, and $g_{xy} = -3$, so $D = 12y - 9$. Testing at $(0,0)$, we get $D = -9 < 0$, so $(0,0)$ will be a saddle point. Testing at $(\frac{9}{4}, \frac{3}{2})$, we get $D = 9 > 0$, so $(\frac{9}{4}, \frac{3}{2})$ is some manner of extremum; using the traditional second-derivative test, we know $g_{xx} > 0$, so it is specifically a local minimum.

4. **(5 points)** *If $x = e^{t+s}$ and $y = ts - 4t$, and $u = x^2 + y^3$, find $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial s}$; your answers need not be algebraically simplified.*

   We may note, for later invocation, that $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 3y^2$, $\frac{\partial x}{\partial t} = e^{t+s}$, $\frac{\partial x}{\partial s} = e^{t+s}$, $\frac{\partial y}{\partial t} = s - 4$, and $\frac{\partial y}{\partial s} = t$. With these simple partials determined, invocation of the chain rule is easy:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = (2x)(e^{t+s}) + (3y^2)(s-4)$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = (2x)(e^{t+s}) + (3y^2)t$$

5. **(2 point bonus)** *Let $\mathbf{r}(t)$ be a vector-valued function describing a curve in space. Prove that if $\nabla f(x,y,z)$ is perpendicular to $\mathbf{r}'(t)$ at every point on the curve given by $\langle x,y,z\rangle = \mathbf{r}(t)$, then $f(x,y,z)$ is constant on the curve.*

We are given perpendicularity of $\nabla f(\mathbf{r}(t))$ and $\mathbf{r}'(t)$ for all t; we may thus assert that for all t, if $\langle x,y,z\rangle = \mathbf{r}(t)$, then $\nabla f(x,y,z)\cdot\mathbf{r}'(t) = 0$.

Now, let us consider $\frac{d}{dt}f(x,y,z)$ where $\langle x,y,z\rangle = \mathbf{r}(t)$. Using the chain rule:

$$\begin{aligned}\frac{d}{dt}f(x,y,z) &= f_x(x,y,z)\frac{dx}{dt} + f_y(x,y,z)\frac{dy}{dt} + f_z(x,y,z)\frac{dz}{dt}\\ &= \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle\cdot\langle\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\rangle\\ &= \nabla f(x,y,z)\cdot\mathbf{r}'(t) = 0\end{aligned}$$

so as (x,y,z) moves along the curve parameterized by t, the derivative of $f(x,y,z)$ with respect to t is zero, so $f(x,y,z)$ is constant for all values of t; thus, $f(x,y,z)$ is constant on the curve described by $\langle x,y,z\rangle = \mathbf{r}(t)$.