1. **(5 points)** *Calculate the integral $\iiint_E y dV$, where E is bounded by the planes $x = 0$, $y = 0$, $y = 1$, $z = 0$, and $2x + z = 4$.*

   This shape can be described by the inequalities $0 \le y \le 1$, $0 \le x$, $0 \le z$, and $2x + z \le 4$. The last of these can be turned into $z \le 4 - 2x$, and sine $z \ge 0$, it is also necessary that $x \le 2$, so our boundaries can be stated as $0 \le x \le 2$, $0 \le y \le 1$, and $0 \le z \le 4 - 2x$, and thus the integral, evaluated in Cartesian coordinates, is:

$$\begin{aligned}\iiint_E y dV &= \int_0^2 \int_0^1 \int_0^{4-2x} y dz dy dx \\ &= \int_0^2 \int_0^1 yz\Big]_{z=0}^{z=4-2x} dy dx = \int_0^2 \int_0^1 4y - 2xy dy dx \\ &= \int_0^2 2y^2 - xy^2\Big]_{y=0}^{y=1} dy dx = \int_0^2 2 - x dx \\ &= 2x - \frac{1}{2}x^2\Big]_0^2 = 2\end{aligned}$$

2. **(5 points)** *Set up (but do not evaluate) a cylindrical form for the integral $\iiint_E (x+1) dV$ over the solid which lies below the paraboloid $z = 8 - x^2 - y^2$ and above the paraboloid $z = x^2 + y^2$.*

   These paraboloids intersect when $8 - x^2 - y^2 = x^2 + y^2$; that is, when $x^2 + y^2 = 4$, or in the parlance of polar coordinates, when $r^2 = 4$. So our range of r-values relevant for describing this solid is $[0, 2]$. The solid is radially symmetric, since both of the paraboloids described are radially symmetric, and thus the standard full interval $0 \le \theta \le 2\pi$ suffices. The lower surface is described by $z = r^2$, and the upper by $z = 8 - r^2$, so the bounds on z are $r^2 \le z \le 8 - r^2$, and so we can now express the integral in cylindrical notation:

$$\iiint_E (x+1) dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} (r\cos\theta + 1) r dz dr d\theta$$

   This evaluates, incidentally, to 16π.

3. **(5 points)** *Set up (but do not evaluate) a spherical integral to calculate the volume lying between the cones $z^2 = x^2 + y^2$ and $z^2 = 3x^2 + 3y^2$ for $z \ge 0$, and bounded above by the sphere $x^2 + y^2 + z^2 = 16$.*

   The sphere $x^2 + y^2 + z^2 = 16$ can be converted, of course, to $\rho^2 = 16$ or $\rho = 4$. The cone $z^2 = x^2 + y^2$ translates to $\rho^2 \cos^2\phi = \rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta$, which can be converted via simple algebra and trigonometry to $\tan^2\phi = 1$, or $\tan\phi = \pm 1$; since we are only looknig at cases with $z > 0$, the negative tangent possibilities can be discarded, and the half-cone that remains corresponds to $\phi = \arctan 1 = \frac{\pi}{4}$.

   Using an almost identical line of argument, the cone $z^2 = 3x^2 + 3y^2$ has spherical representation $\tan^2\phi = \frac{1}{3}$, so $\phi = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$. Thus, the solid between these two cones is described by the inequality $\frac{\pi}{6} \le \phi \le \frac{\pi}{4}$. We thus have boundaries on both ρ and ϕ, and, since this solid is radially symmetric, θ can take on the usual unrestricted limits to give the volume integral:

$$\iiint_E dV = \int_0^{2\pi} \int_{\pi/6}^{\pi/4} \int_0^4 \rho^2 \sin\phi d\rho d\phi d\theta$$

   This integral, incidentally, is $\frac{64\pi(\sqrt{3}-\sqrt{2})}{3}$.

4. **(5 points)** *Let the region D be bounded by the curves $xy = 1$, $xy = 4$, $y = 4x$, and $y = 9x$ in the first quadrant. Use the transformations $x = \frac{u}{v}$ and $y = uv$ to compute the area of D.*

Note that the boundaries are simple in u and v: $xy = 1$ becomes $(\frac{u}{v}(uv) = u^2 = 1$, and since in the first quadrant u and v will be positive, $u = 1$; liewise $xy == 4$ becomes $u^2 = 4$ or $u = 2$. In addition, $y = 4x$ translates to $uv = 4\frac{u}{v}$, so $v^2 = 4$ or $v = 2$, and $y = 9x$ becomes $v^2 = 9$ or $v = 3$. Thus, the area of D is simply enough expressed as a uv-integral:

$$\iint_D dA = \int_1^2 \int_2^3 \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dvdu$$

but we must calculate

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}$$

And then the integral itself is not too hard:

$$\int_1^2 \int_2^3 \frac{2u}{v} dvdu = \int_1^2 2u \ln |v|\Big]_{v=2}^{v=3} du = \int_1^2 2u(\ln 3 - \ln 2)du = (\ln \frac{3}{2})u^2\Big]_1^2 = 3 \ln \frac{3}{2}$$