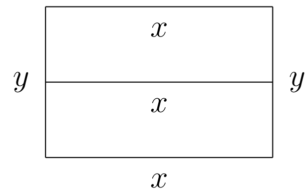
1. **(10 points)** *You are placing a fence around all four sides of a farmyard, as well as a fence parallel to the front fence running down the middle. You have 600 feet of fencing.*

   (a) **(4 points)** *Find the length of the side fences as a function of the length of the front fence.*

   Above we have drawn this figure, with x denoting the length of the front fence, and y the side-fence length. Adding up all the fences shown, it is clear that this figure uses a length $3x + 2y$ of fencing; since we have 600 feet of fence to use in total, it is thus necessary that $3x + 2y = 600$. Then, expressing y as a function of x, we would rearrange the above to give $2y = 600 - 3x$, and then $y = \frac{600-3x}{2}$.

   (b) **(2 points)** *Find the area of the yard as a function of the length of the front fence.*

   Since the area of the field drawn above is xy, and since we jsut determined that $y = \frac{600-3x}{2}$, the function describing the area is $A(x) = x\left(\frac{600-3x}{2}\right) = \frac{-3}{2}x^2 + 300x$.

   (c) **(4 points)** *Find the dimensions of the yard which maximize its area.*

   We seek the maximum of $A(x)$ and, more to the point, the value of x yielding that maximum. To do so, we may complete the square to put $A(x)$ in standard form (or use any other vertex-identifying procedure):

$$\begin{aligned}
A(x) &= \frac{-3}{2}x^2 + 300x \\
&= \frac{-3}{2}x^2 + 300x + \frac{(300)^2}{4 \cdot \frac{-3}{2}} - \frac{(300)^2}{4 \cdot \frac{-3}{2}} \\
&= \frac{-3}{2}x^2 + 300x + \frac{90000}{-6} - \frac{90000}{-6} \\
&= \frac{-3}{2}\left(x^2 - 200x + 10000\right) + 15000 \\
&= \frac{-3}{2}(x - 100)^2 + 15000
\end{aligned}$$

   so $A(x)$ is maximized when $x = 100$, and specifically $A(100) = 15000$. Since the dimensions were asked for, we need x *and* y: $y = \frac{600-3x}{2} = 150$. So the field is $100' \times 150'$.

2. **(15 points)** *Perform the following arithmetic and algebraic operations.*

   (a) **(3 points)** *Simplify the expression* $\frac{x}{x-1} + \frac{2}{x-2}$.

We find a common denominator and simplify the numerator:

$$\begin{aligned}\frac{x}{x-1}+\frac{2}{x-2} &= \frac{x(x-2)}{(x-1)(x-2)}+\frac{(x-1)2}{(x-1)(x-2)}\\ &= \frac{x(x-2)+(x-1)2}{(x-1)(x-2)}\\ &= \frac{x^2-2x+2x-2}{x^2-3x+2}\\ &= \frac{x^2-2}{x^2-3x+2}\end{aligned}$$

The denominator could be left factored; expansion versus factorization is a stylistic variation within the bounds of what is considered "simplified".

(b) **(3 points)** *Calculate* $27^{-4/3}$.

Exploding the rational exponent into three individually calculable steps, we get:

$$27^{-4/3} = \left(\left(27^{1/3}\right)^4\right)^{-1} = \left(\sqrt[3]{27}^4\right)^{-1} = \frac{1}{3^4} = \frac{1}{81}$$

(c) **(3 points)** *Factor the quadratic* x^2-7x-8.

Trial and error suggests the possibile factorizations $(x-1)(x+8)$, $(x-2)(x+4)$, $(x-4)(x+2)$, and $(x-8)(x+1)$. The last of these is correct. This can also be found using the quadratic formula to note that x^2-7x-8 is zero when $x=8$ and $x=-1$.

(d) **(3 points)** *Expand and simplify the polynomial* $(x^2+1)(2x-4)-(x^3+2)$.

Using straightforward algebraic techniques:

$$\begin{aligned}(x^2+1)(2x-4)-(x^3+2) &= (2x^3-4x^2+2x-4)-(x^3+2)\\ &= x^3-4x^2+2x-6\end{aligned}$$

(e) **(3 points)** *Simplify the expression* $\left(\frac{x^3y^2}{z}\right)^4\left(\frac{xz^2}{y^3}\right)$.

Using exponential distribution-and-gathering techniques:

$$\left(\frac{x^3y^2}{z}\right)^4\left(\frac{xz^2}{y^3}\right) = \frac{(x^3)^4(y^2)^4}{z^4}\left(\frac{xz^2}{y^3}\right) = \frac{x^{12}y^8}{z^4}\cdot\frac{xz^2}{y^3} = \frac{x^{13}y^5}{z^2}$$

3. **(15 points)** *Answer the following questions about the functions* $f(x)=\frac{2}{x+4}$ *and* $g(x)=x^2-9$.

(a) **(3 points)** *Determine the domains of* $f(x)$ *and* $g(x)$.

$f(x)$'s domain must exclude those points where the denominator of the fraction is zero: thus its domain consists of those values where $x+4\neq 0$, or $x\neq -4$.

There are no impediments whatsoever to calculating $g(x)$, so its domain consists of allr eal x.

(b) **(2 points)** *Write formulas, which need not be simplified, for* $(f-g)(x)$ *and* $\frac{f}{g}(x)$.

$(f-g)(x) = f(x) - g(x) = \frac{2}{x+4} - (x^2-9)$, and $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{\frac{2}{x+4}}{x^2-9}$.

If you insist on simplifying these expressions, you will get $\frac{-x^3-4x^2+9x+38}{x+4}$ and $\frac{2}{(x+4)(x^2-9)}$ respectively.

(c) **(4 points)** *Determine the domains of* $(f-g)(x)$ *and* $\frac{f}{g}(x)$.

The domain of $(f-g)(x)$ consists of the overlap of the domains of $f(x)$ and $g(x)$. $g(x)$'s domain is all real numbers; $f(x)$'s consists of all real numbers except -4; they overlap on all x such that $x \neq -4$.

To calculate the domain of $\frac{f}{g}(x)$, we begin with the same restrictions as above, and in addition requare that $g(x) \neq 0$. In order for $x^2 - 9$ to be nonzero, it must be the case that $x^2 \neq 9$, so $x \neq \pm 3$. Thus, our complete set of domain restrictions on $\frac{f}{g}(x)$ is $x \neq -4$, $x \neq -3$, and $x \neq 3$.

(d) **(3 points)** *Write formulas, which need not be simplified, for* $f(g(x))$ *and* $g(g(x))$.

$$f(g(x)) = f(x^2-9) = \frac{1}{(x^2-9)+4} = \frac{1}{x^2-5}$$
$$g(g(x)) = g(x^2-9) = (x^2-9)^2 - 9 = x^4 - 18x^2 + 72$$

The last step in each of these calculations is a simplification and may be omittted.

(e) **(3 points)** *Find the inverse of the function* $f(x)$.

$$\begin{aligned}
f(f^{-1}(x)) &= x \\
\frac{1}{f^{-1}(x)+4} &= x \\
1 &= x(f^{-1}(x)+4) \\
\frac{1}{x} &= f^{-1}(x) + 4 \\
\frac{1}{x} - 4 &= f^{-1}(x)
\end{aligned}$$

so the inverse function of $f(x)$ is $f^{-1}(x) = \frac{1}{x} - 4$.

4. **(10 points)** *Answer the following questions about the quadratic* $s(x) = -3x^2 - 12x - 12$.

(a) **(3 points)** *Put the quadratic* $s(x)$ *in standard form.*

The square-completion term we calculate is $\frac{b^2}{4a} = \frac{144}{-12} = -12$, which is indeed already present in the quadratic, so we factor out -3 and hope for a square, which we in fact get:

$$\begin{aligned}
s(x) &= -3x^2 - 12x - 12 \\
&= -3(x^2 + 4x + 4) \\
&= -3(x+2)^2 + 0
\end{aligned}$$

The $+0$ on the end is an unnecessary formalism to exhibit that this expression does in fact have the structure $a(x-h)^2 + k$.

(b) **(1 point)** *Does this function have a maximum or minimum value? If so, identify which it is and what its value is.*

The vertex of this quadratic is $(-2, 0)$. Since the coefficient -3 on the quadratic is negative, it opens downwards. Thus the value $s(-2) = 0$ is the maximum of this function.

(c) **(4 points)** *Determine its vertex, x-intercepts if they exist, and y-intercept.*

The vertex, as described above, is $(-2, 0)$. This is also clearly the x-intercept (although that information could also be determined by applying the quadratic formula). To find the y-intercept, we evaluate $s(0) = -12$, so the y-intercept is $(0, -12)$.

(d) **(2 points)** *What is the average rate of change of the function $s(x)$ between the points $x = 0$ and $x = 2$?*

The rate of change is calculated as follows:

$$\begin{aligned}\frac{s(2) - s(0)}{2 - 0} &= \frac{(-3 \cdot 2^2 - 12 \cdot 2 - 12) - (-3 \cdot 0^2 - 12 \cdot 0 - 12)}{2 - 0} \\ &= \frac{(-12 - 24 - 12) - (-12)}{2} = \frac{-36}{2} = -18\end{aligned}$$

5. **(10 points)** Answer the following questions about graphs.

(a) **(2 points)** *Determine the equation of the line through the point $(-1, 2)$ which is perpendicular to $y = 2x + 4$.*

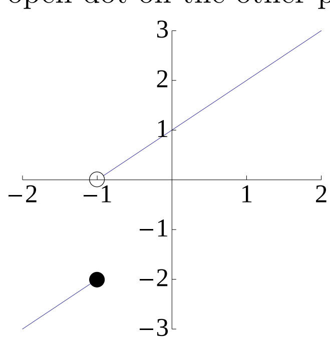
Since it is perpendicular to a line of slope 2, this line should have slope $\frac{-1}{2}$. And, since it passes through $(-1, 2)$, we have enough information now to express its equation in point-slope form:

$$(y - 2) = \frac{-1}{2}(x + 1)$$

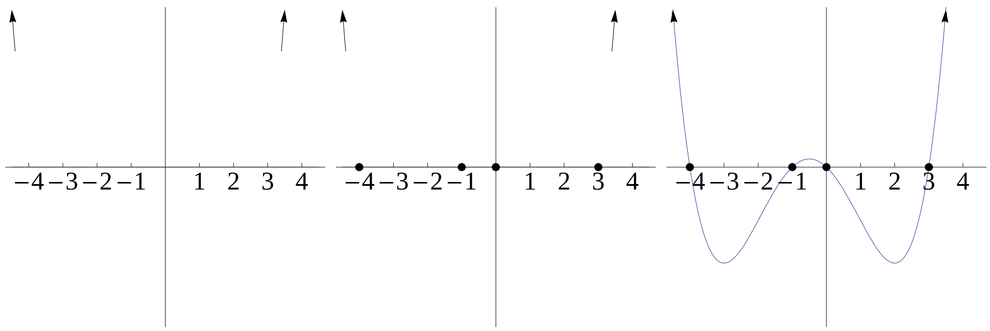
Some people may prefer slope-intercept form, in which the above equation becomes $y = \frac{-1}{2}x + \frac{3}{2}$.

(b) **(2 points)** *Draw the graph of the piecewise function* $g(x) = \begin{cases} x - 1 \text{ if } & x \le -1 \\ x + 1 \text{ if } & x > -1 \end{cases}$

We draw the equation of the line $x - 1$ to the left of the value $x = -1$, and $x + 1$ to the right of $x = -1$. Since $g(-1) = -1 - 1 = -2$, we put a solid dot at $(-1, -2)$, and an open dot on the other piece of the function at the same x-value.



(c) **(3 points)** *Sketch the graph of the function $f(x) = 2x(x - 3)(x + 4)(x + 1)$.*

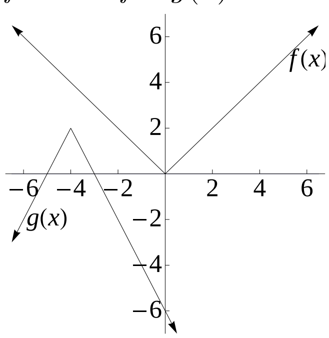


The leading term of this on expansion would be $2x^4$, so we know the long-term behavior of this function has upwards-facing arrows on the left and right sides, as befits a polynomial whose leading term has an even exponent and a positive coefficient. This is shown on the first picture above.

Now we identify zeroes. The factorization above makes it clear that $f(-4) = f(-1) = f(0) = f(3) = 0$, so we draw in all those points, on the second picture.

Finally, we can connect the dots, as seen on the third picture above.

(d) **(3 points)** *The graph* $f(x) = |x|$ *is shown, together with a transformation* $g(x)$*. Find a formula for* $g(x)$*.*



There is a shift, a flip, and a stretch in this transformation. We might start by performing the vertical flip and a vertical stretch by 2, to get a partial transform $-2f(x)$ (note: we could also interpret the reshaping as a horizontal squashing by a factor of $\frac{1}{2}$, which would be $-f(2x)$; these are actually the same function in this particular case). Now, we need to need to shift left by 4 units and up by 2: we thus get $g(x) = -2f(x+4)+2 = -2|x+4|+2$. Alternatively, we might get $-|2(x+4)|+2$, if interpreting the reshaping as a horizontal rather than vertical deformation.