1. **(10 points)** *Answer the following questions about growth and decay.*

   (a) **(3 points)** *Market analysis suggests than an investment in Tessier-Ashpool S.A. will have a relative growth rate of 6% per year. If you invest \$5000 in this corporation, how long will it take your investment to grow to a value of \$6000?*

   The model for the growth of a \$5000 investment with a 6% rate of growth is $f(t) = 5000e^{0.06t}$, and here we are asking which value of t yields $f(t) = 6000$. Thus:

$$\begin{aligned} 5000e^{0.06t} &= 6000 \\ e^{0.06t} &= \frac{6000}{5000} = \frac{6}{5} \\ 0.06t &= \ln\frac{6}{5} \\ t &= \frac{\ln\frac{6}{5}}{0.06} \approx 3.04 \text{ years} \end{aligned}$$

   The approximation step is provided for context, and need not be calculated under exam conditions.

   (b) **(3 points)** *Miraclo is a drug with a half-life in the body of 5 hours. Produce a function describing the quantity still in the body t hours after administration of a 40mg dose.*

   A decay function with a given half-life τ is given by either $f(t) = A\left(\frac{1}{2}\right)^{\frac{t}{\tau}}$ or $f(t) = Ae^{\frac{-\ln 2}{\tau}t}$, so in this case we would produce the function $f(t) = 40\left(\frac{1}{2}\right)^{\frac{t}{5}}$ or $f(t) = 40e^{\frac{-\ln 2}{5}t}$. Note that these two functions are actually algebraically equivalent.

   Alternatively, if one dislikes appeal to the half-life formula, one can start with an arbitrary exponential function template $f(t) = Ae^{rt}$ and use the fact $f(0) = 40$ to find $A = 40$, and the fact $f(5) = 20$ to find:

$$\begin{aligned} 20 = f(5) &= 40e^{r\cdot 5} \\ \frac{1}{2} &= e^{5r} \\ \ln\frac{1}{2} &= 5r \\ \frac{1}{5}\ln\frac{1}{2} &= r \end{aligned}$$

   (c) **(2 points)** *Miraclo becomes ineffective when there is less than 15mg in the body. Using your result from the above question, determine how long after taking a 40mg dose this will occur.*

   From the above formula in exponential form, solving for $f(t) = 15$, we get

$$\begin{aligned} 40e^{\frac{-\ln 2}{5}t} &= 15 \\ e^{\frac{-\ln 2}{5}t} &= \frac{15}{40} = \frac{3}{8} \\ \frac{-\ln 2}{5}t &= \ln\frac{3}{8} \\ t &= \frac{-5\ln\frac{3}{8}}{\ln 2} \approx 7.07 \text{ hours} \end{aligned}$$

The approximation step is provided for context, and need not be calculated under exam conditions.

(d) **(2 points)** *An alien spacecraft, red-hot from its entry into the atmosphere, lands on a warm summer day. Its temperature in degrees Fahrenheit t minutes after impact is given by the function $f(t) = 80 + 1600e^{-0.015t}$. A science team can begin experiments on it after it has cooled to $400°F$. How many minutes will they need to wait?*

We are solving for the value of t such that $f(t) = 400$:

$$\begin{aligned} 80 + 1600e^{-0.015t} &= 400 \\ 1600e^{-0.015t} &= 320 \\ e^{-0.015t} &= \frac{320}{1600} = \frac{1}{5} \\ -0.015t &= \ln\frac{1}{5} \\ t &= \frac{\ln\frac{1}{5}}{-0.015} \approx 107.3 \text{ minutes} \end{aligned}$$

The approximation step is provided for context, and need not be calculated under exam conditions.

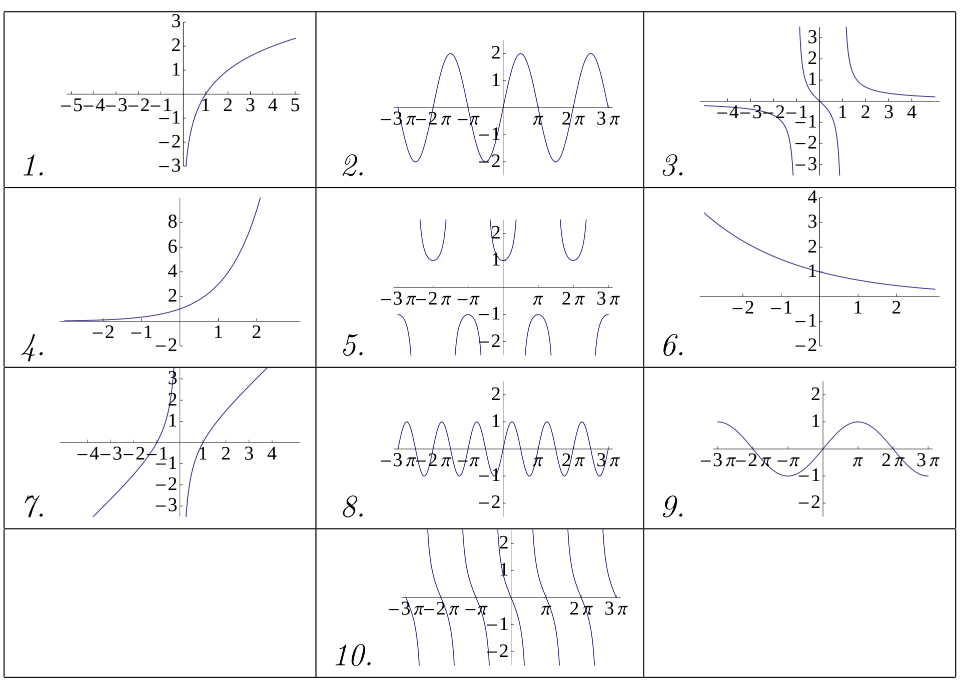
2. **(10 points)** *The following ten graphs are of the following functions:*

$$A(x) = 3^x \qquad B(x) = \left(\frac{2}{3}\right)^x \qquad C(x) = \log_2 x \qquad D(x) = \sin 2x$$

$$E(x) = 2\sin x \qquad F(x) = \sin\frac{x}{2} \qquad G(x) = \tan(-x) \qquad H(x) = \sec x$$

$$I(x) = \frac{x}{x^2-1} \qquad J(x) = \frac{x^2-1}{x}$$

Label each picture with the letter of the appropriate function.



Several of the graphs have highly characteristic and unique shapes which we can identify merely on sight. Graph #1 has an asymptote at 0 and exists only in the positive half of the plane; this is characteristic of a logarithm so it must be $C(x)$. Graph #5 has the disjoint humps and central gap of a secant-like function, so it must be $H(x)$. Graph #10 has the snaky repetition of a tangent-like function, and thus must be $G(x)$.

Of the remainder, we can broadly classify then into sinusoidal, exponential, and "other". Graphs #2, #8, and #9 are sinusoidal and must thus correspond in some order to $D(x)$, $E(x)$, and $F(x)$. Graph #2 has amplitude of 2, which corresponds to a multiplication by 2 outside of the trig function, as appears in $E(x)$. Graph #8 has period of π and graph 9 has period of 4π, so they are respectively the result of a horizontal compression and a stretch of the ordinary sine function: we see that $D(x)$ is a compression and $F(x)$ is a stretch.

Now we handle the exponentials, of which there are two: graph #4 resembles a growth curve while graph #6 is a decay curve. On the functional side, there is $A(x)$, which has a base larger than 1 and thus is a growth curve, while $B(x)$ has a base less than 1 and represents decay.

Finally, the two oddities are graphs #3 and #7, which we recognize as corresponding in some order to the rational functions $I(x)$ and $J(x)$. The easiest visual identification is by vertical asymptotes; $I(x)$ has $x^2 - 1$ in the denominator and must have asymptotes at $x = -1$ and $x = 1$, as graph #3 does, while $J(x)$ has x in the denominator and has an asymptote at $x = 0$, like graph #7.

3. **(15 points)** *Answer the following questions about trigonometry.*

   (a) **(2 points)** *Evaluate* $\csc \frac{17\pi}{6}$.

   Note that $\frac{17\pi}{6} = (2\frac{5}{6})\pi$, so $\csc \frac{17\pi}{6} = \csc \frac{5\pi}{6} = \frac{1}{\sin \frac{5\pi}{6}}$. The angle $frac5\pi6$ describes a point in the second quadrant, in which the sine in positive, so $\sin \frac{5\pi}{6}$ is positive, and since this argument is a fully reduced fraction of the form $\frac{k\pi}{6}$, we know that $\sin \frac{5\pi}{6} = \pm\frac{1}{2}$; since it is positive, we know $\csc \frac{17\pi}{6} = \frac{1}{1/2} = 2$.

   (b) **(2 points)** *Evaluate* $\sin 270°$.

   We know that $270°$ is $\frac{2\pi}{360} \cdot 270 = \frac{3\pi}{2}$ radians. The sine of any odd multiple of $\frac{\pi}{2}$ is ± 1, and in this case, since an angle of $\frac{3\pi}{2}$ describes a point on the negative y-axis, this sine is -1.

   (c) **(3 points)** *Identify the period, amplitude, and vertical shift of the periodic function* $g(x) = \frac{1}{3}\cos(\frac{3}{5}x) + 8$.

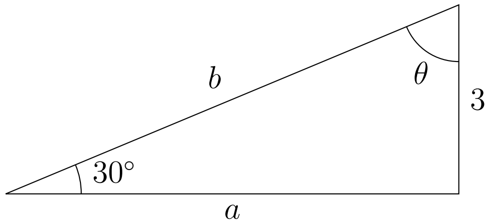
   The vertical shift can be clearly seen to be $+8$, since the addition of 8 to the function value enacts a vertical shift of 8 on the graph.

   The amplitude and period are dictated by the vertical and horizontal stretch factors respectively: we see that this graph is vertically stretched by a factor of $\frac{1}{3}$ (i.e. compressed by a factor of 3), so the amplitude of the standard cosine function, which is 1, is transformed to $\frac{1}{3}$.

   The period is dictated by the compression factor indicated in the argument to the cosine. We compress the ordinary cosine curve horizontally by a factor of $\frac{3}{5}$, so the standard period of 2π is transformed into a period of length $\frac{2\pi}{3/5} = \frac{10\pi}{3}$.

   (d) **(3 points)** *If* $\sec x = \frac{-13}{12}$ *and* x *describes a terminal point in the second quadrant, what is* $\cot x$*?*

(e) **(3 points)** *In the following right triangle (not drawn to scale), find the values of a, b, and θ. Label which is which.*

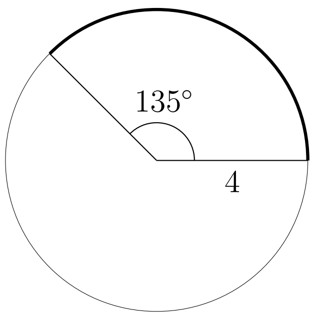


We know the angles of a triangle add up to 180°, so $30 + 90 + \theta = 180$, which simplifies to $\theta = 60$.

Using known trigonometric values for 30° and their relationship to triangle sides, we see that $\frac{3}{b} = \sin(30°) = \frac{1}{2}$. Simplifying this algebraic expression, we will find that $b = 6$.

Using another known trigonometric value, we see that $\frac{a}{b} = \cos(30°) = \frac{\sqrt{3}}{2}$. Since we know $b = 6$, we find that $a = \frac{\sqrt{3}}{2}b = 3\sqrt{3}$.

(f) **(2 points)** *Find the length of the indicated arc (not drawn to scale).*



We know that the length of an arc of radius r and angle θ is $r\theta$ if θ is given in radians, which this arc is not. So first, we shall convert it: 135° is $135 \cdot \frac{2\pi}{360} = \frac{3}{4}\pi$ radians. And now, the length of the arc will be $4 \cdot \frac{3}{4}\pi = 3\pi$.

4. **(5 points)** *Answer the following questions about complex numbers.*

(a) **(3 points)** *Express $(2+3i)(1-i) - (1+i)(4i)$ in the form $a + bi$.*

We evaluate each of the products via the distributive law, and make use of the fact that $i^2 = -1$, and then simplify:

$$\begin{aligned}(2+3i)(1-i) - (1+i)(4i) &= (2 - 2i + 3i - 3i^2) - (4i + 4i^2) \\ &= (2 - 2i + 3i + 3) - (4i - 4) = 9 - 3i\end{aligned}$$

(b) **(2 points)** *Express $\frac{1+2i}{3-i}$ in the form $a + bi$.*

We multiply the numerator and denominator by the conjugate $(3 + i)$ to get the denominator to be noncomplex:

$$\begin{aligned}\frac{1+2i}{3-i} &= \frac{(1+2i)(3+i)}{(3-i)(3+i)} \\ &= \frac{3 + i + 6i + 2i^2}{9 + 3i - 3i - i^2} = \frac{1+7i}{10} = \frac{1}{10} + \frac{7}{10}i\end{aligned}$$

5. **(10 points)** *Answer the following questions concerning logarithms.*

(a) **(2 points)** *Express* $5\ln(xy^2) - \frac{1}{4}\ln x - 3\ln(\frac{yz^3}{x})$ *as a single logarithm.*
Using the rule $n\log_b x = \log_b(x^n)$, we can simplify this expression to

$$\ln(xy^2)^5 - \ln x^{1/4} - \ln\left(\frac{yz^3}{x}\right)^3 = \ln(x^5y^{10}) - \ln x^{1/4} - \ln\frac{y^3z^9}{x^3}$$

and then, using the rule $\ln a - \ln b = \ln\frac{a}{b}$ we shall convert this to

$$\ln\frac{x^5y^{10}}{x^{1/4}\frac{y^3z^9}{x^3}} = \ln\frac{x^{31/4}y^7}{z^9}$$

(b) **(3 points)** *Find the domain of the function* $f(x) = \frac{x^2-9}{\log_{10}(x-1)}$.
We have two potential problems: the denominator could be zero, or the argument of the logarithm could be nonpositive. We thus require that $x-1 > 0$ and $\log_{10}(x-1) \neq 0$. The first condition is easily found to be $x > 1$; the second, in contrast, is that $x-1 \neq 10^0 = 1$, or $x \neq 2$. Thus, the domain consists of values $x > 1$ and $x \neq 2$; in interval notation, x lies in one of the intervals $(1, 2)$ or $(2, +\infty)$.

(c) **(3 points)** *Determine the value of* $\log_5 30 - \frac{1}{2}\log_5 144 + \log_5 10$.
Using our logarithm-simplification rules,

$$\begin{aligned}\log_5 30 - \frac{1}{2}\log_5 144 + \log_5 10 &= \log_5 30 - \log_5\sqrt{144} + \log_5 10\\ &= \log_5\frac{30\cdot 10}{\sqrt{144}} = \log_5\frac{300}{12} = \log_5 25 = 2\end{aligned}$$

(d) **(2 points)** *Solve for* x *in the exponential equation* $3\cdot 2^{3x-1} = 48$.
Isolating the x algebraically, we see:

$$\begin{aligned}3\cdot 2^{3x-1} &= 48\\ 2^{3x-1} &= \frac{48}{3} = 16\\ 3x-1 &= \log_2 16 = 4\\ 3x &= 5\\ x &= \frac{5}{3}\end{aligned}$$

6. **(10 points)** *Answer the following questions preparatory to sketching the rational function* $h(x) = \frac{3(x+2)(x-1)}{x(x+4)}$.

(a) **(2 points)** *What is the function's domain?*
The denominator cannot be zero, so $x \neq 0$ and $x + 4 \neq 0$, which simplifies to $x \neq 0, -4$. In interval form, this would be $(-\infty, 0), (0, 4), (4, \infty)$.

(b) **(2 points)** *Does this function have* x*-intercepts, and if so, what are they?*
$h(x) = 0$ when its numerator is zero. $3(x + 2)(x - 1) = 0$ when $x + 2 = 0$ or $x - 1 = 0$. Thus, $h(x)$ has the x-intercepts -2 and 1.

(c) **(2 points)** *Where are this function's vertical asymptotes?*

These are precisely the places where the denominator becomes zero, precipitating a "blow-up" of the function. As seen above, these points are $x = 0$ and $x = 4$.

(d) **(3 points)** *How does this function behave as x becomes very large? How does it behave as x becomes very highly negative? Label which is which.*

This function is, in the long term, behaves much like $\frac{3x^2}{x^2} = 3$. So if we were to look at $h(x)$ at very large values of x, $h(x)$ will be very close to 3; likewise, for x very highly negative, $h(x)$ is close to 3.

(e) **(1 point)** *Does this function have a maximum or minimum value? Why or why not?*

It has neither; we can get arbitrarily large numbers or arbitrarily low numbers by considering x-values close to the asymptotes, e.g. $h(3.999)$ of $h(4.001)$.