1. **(10 points)** *Answer the following questions about evaluating trigonometric expressions.*

   (a) **(3 points)** *Evaluate the expression* $\sin(75°)\cos(15°) - \cos(75°)\sin(15°)$.

   The above expression is a template which is recognizable as one of those covered in our angle-addition rules; recall that

   $$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

   whose right side matches nicely with the expression seen here, so

   $$\sin(75°)\cos(15°) - \cos(75°)\sin(15°) = \sin(75° - 15°) = \sin 60° = \frac{\sqrt{3}}{2}$$

   (b) **(2 points)** *Evaluate* $\arccos \frac{-\sqrt{2}}{2}$.

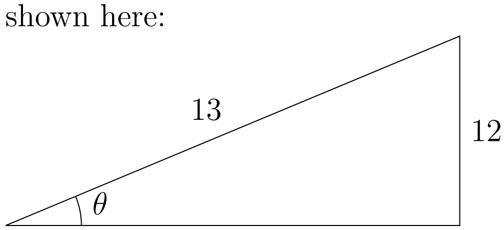
   There are a few familiar cosines, and their associated arccosines should be discernable from the familiarity of their form. Recall that the arccosine function returns values between 0 and $\pi$. We know that $\cos \frac{3\pi}{4} = \frac{-\sqrt{2}}{2}$, and thus $\arccos \frac{-\sqrt{2}}{2} = \frac{3\pi}{4}$.

   (c) **(2 points)** *Evaluate* $\arctan \frac{1}{\sqrt{3}}$.

   There are a few familiar tangents, and their associated arctangents should be discernable from the familiarity of their form. For instance, we know that $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$, and thus $\arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}$.

   (d) **(4 points)** *Evaluate* $\tan \arcsin(\frac{12}{13})$.

   We give a simple name to $\arcsin \frac{12}{13}$; traditionally we might call it $\theta$. To convey this information we might build a triangle exemplifying this relationship between $\theta$ and $\frac{12}{13}$, which we might write as $\theta = \arcsin \frac{12}{13}$, but more comprehensibly as $\sin\theta = \frac{12}{13}$; in a right triangle with $\theta$ as one of the angles, we know that $\sin\theta$ represents the ratio of the opposite side and the hypotenuse. We would thus represent this relationship by making the opposite side of the triangle have length 12, and the hypotenuse have length 13, as shown here:




   Furthermore, the adjacent side, which is not labeled in the above picture, can be calculated by the Pythagorean Theorem to have length $\sqrt{13^2 - 12^2} = \sqrt{169 - 144} = \sqrt{25} = 5$. After that setup, our actual calculation ends up being very easy! We wanted to find $\tan \arcsin \frac{12}{13}$, which in light of our definition of $\theta$ can be written as simply $\tan\theta$. Finding any trigonometric identity of an angle in a labeled right triangle is simply a matter of dividing the appropriate sides: the tangent is the ratio of the opposite and adjacent sides, so in this case, $\tan\theta = \frac{12}{5}$.

2. **(8 points)** *Identify each of the following sequences as arithmetic, geometric, or neither, and find a formula for each sequence.*

(a) **(2 points)** $5, 10, 20, 40, 80, \ldots$.

Note that the differences of the 4 pairs of consecutive terms seen here are 5, 10, 20, and 40; these are not the same, so this sequence is not arithmetic. The ratios of the 4 pairs of consecutive terms seen here, however, are 2, 2, 2, and 2; since these are the same, this sequence is geometric with common ratio 2 and first term 5, and thus has formula $a_n = 5 \cdot 2^{n-1}$.

(b) **(2 points)** $7, 5, 3, 1, -1, \ldots$.

Note that the differences of the 4 pairs of consecutive terms seen here are $-2$, $-2$, $-2$, and $-2$; since these are the same, this sequence is arithmetic with common difference $-2$ and first term 7, and thus has formula $a_n = 7 - 2(n-1)$.

(c) **(2 points)** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$.

Note that the differences of the 4 pairs of consecutive terms seen here are $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{20}$, and $\frac{1}{30}$; since these are the not the same, this sequence is not arithmetic. The ratios of the 4 pairs of consecutive terms seen here are $\frac{4}{3}$, $\frac{9}{8}$, $\frac{16}{15}$, and $\frac{25}{24}$; since these are not the same, this sequence is not geometric. This is thus a sequence which is neither arithmetic nor geometric, but inspection of the pattern will reveal it to be given by the formula $a_n = \frac{n}{n+1}$.

(d) **(2 points)** $-4, 10, -25, \frac{125}{2}, \frac{-625}{4}, \ldots$.

Note that the differences of the 4 pairs of consecutive terms seen here are 14, $-35$, $\frac{175}{2}$, and $\frac{-875}{4}$; these are not the same, so this sequence is not arithmetic. The ratios of the 4 pairs of consecutive terms seen here, however, are $\frac{-5}{2}$, $\frac{-5}{2}$, $\frac{-5}{2}$, and $\frac{-5}{2}$; since these are the same, this sequence is geometric with common ratio $\frac{-5}{2}$ and first term $-4$, and thus has formula $a_n = -4\left(\frac{-5}{2}\right)^{n-1}$.

3. **(10 points)** *Answer the following questions about trigonometric equations.*

(a) **(3 points)** *Find* any one *solution to the equation* $3\cos(2x) + 7 = 10$.

Since we were only asked to find one solution, we don't have to be too cautious with our cosine-inversion, and can use the ordinary arc-cosine:

$$\begin{aligned} 3\cos(2x) + 7 &= 10 \\ 3\cos(2x) &= 3 \\ \cos(2x) &= 1 \\ 2x &= \arccos 1 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

There are several other acceptable answers, if a different choice of value whose cosine is 1 is used.

(b) **(3 points)** *Find* all *solutions to the equation* $2\tan(3x) + 6 = 4$.

Note below the procedure used to invert the tangent, to ensure that *all* solutions are found:

$$\begin{aligned}2\tan(3x)+6&=4\\2\tan(3x)&=-2\\\tan(3x)&=-1\\3x&=\arctan(-1)+\pi n\\3x&=\frac{-\pi}{4}+\pi n\\x&=\frac{\pi}{3}n-\frac{\pi}{12}\end{aligned}$$

So $x$ can take on any value of the form $\frac{\pi}{3}n-\frac{\pi}{12}$.

(c) **(4 points)** *Verify the trigonometric identity* $\frac{\sin\theta}{\csc\theta\cos\theta}=\sec\theta-\cos\theta$.

We simplify each side of this equality as far as we can, and explore the extent to which the two simplifications can be converted into each other. The left side is

$$\frac{\sin\theta}{\csc\theta\cos\theta}=\frac{\sin\theta}{\frac{1}{\sin\theta}\cos\theta}=\frac{\sin^2\theta}{\cos\theta}$$

while the right, if we think in terms of expressing it as a fraction with $\cos\theta$ in the denominator, can be manipulated as such:

$$\sec\theta-\cos\theta=\frac{1}{\cos\theta}-\cos\theta=\frac{1-\cos^2\theta}{\cos\theta}$$

and now we can invoke the well-known identity $\sin^2\theta+\cos^2\theta=1$ to bridge the gap between these two expressions:

$$\frac{1-\cos^2\theta}{\cos\theta}=\frac{\sin^2\theta}{\cos\theta}$$

4. **(15 points)** *Answer the following questions about series.*

(a) **(2 points)** *Evaluate* $\sum_{j=2}^{5}2j^2$.

This sum consists of only four terms, and is easily expanded:

$$\sum_{j=2}^{5}2j^2=2(2^2)+2(3^2)+2(4^2)+2(5^2)=8+54+32+50=108$$

(b) **(2 points)** *Express* $\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\cdots+\sqrt{70}$ *in sigma notation.*

We are taking expressions of the form $\sqrt{n}$ and adding them up as $n$ ranges from 1 to 70, so the sigma-notation representation of this sum is $\sum_{n=1}^{70}\sqrt{n}$. Another letter may be used in place of $n$ ($n$, $i$, $j$, and $k$ are traditional, but any letter is valid).

(c) **(3 points)** *Calculate the sum of the first 20 terms of the arithmetic series with first term 5 and common difference* $-3$.

The first term is 5, and we can calculate the twentieth term to be $a_{20} = 5 - 3(20 - 1) = 5 - 57 = -52$. If we use the method famously attributed to Gauss to calculate this sum, we see that

$$\begin{aligned} S &= \quad 5 + \ \ 2 + \cdots - 52 \\ S &= -52 - 49 - \cdots + \ \ 5 \\ 2S &= -47 - 47 - \cdots - 47 = -47(20) \end{aligned}$$

so $S = \frac{(-47)(20)}{2} = -470$.
Alternatively, one can use the formula:

$$S = an + \frac{n(n-1)}{2}d = 5 \cdot 20 + \frac{20(19)}{2}(-3) = 100 - 570 = -470$$

(d) **(3 points)** *Calculate the arithmetic series partial sum* $6 + 8 + 10 + 12 + 14 + \cdots + 104$.
First, we must calculate how many terms this partial sum has. The first term is 6 and the common difference is 2, so the terms are given by the formula $a_n = 6 + 2(n - 1)$. To determine the index on the last term:

$$\begin{aligned} 104 &= 6 + 2(n-1) \\ 98 &= 2(n-1) \\ 49 &= n - 1 \\ 50 &= n \end{aligned}$$

so there are 50 terms in total. We can now find the sum either by the series-reversal trick:

$$\begin{aligned} S &= \ \ 06 + \ \ 08 + \cdots + 104 \\ S &= 104 + 102 + \cdots + \ \ 06 \\ 2S &= 110 + 110 + \cdots + 110 = 50 \cdot 110 \end{aligned}$$

so $S = \frac{50 \cdot 110}{2} = 2750$, or by the sum formula:

$$S = an + \frac{n(n-1)}{2}d = 6 \cdot 50 + \frac{50 \cdot 49}{2}2 = 300 + 2450 = 2750$$

(e) **(3 points)** *Find the geometric series partial sum* $4 - 12 + 36 - 108 + 324 - \cdots + 4 \cdot 3^{10}$. *You may leave* one *unreduced exponent in your answer.*
This series is geometric with common ratio $-3$ and first term 4; we are adding up the first 11 terms (since $4(-3)^{n-1} = 4 \cdot (-3)^{10}$ when $n - 1 = 10$), so we use the formula:

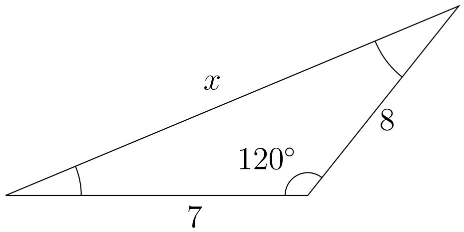
$$S = \frac{a(1 - r^n)}{1 - r} = \frac{4(1 - (-3)^11)}{1 - (-3)} = 1 + 3^{11}$$

(f) **(2 points)** *Evaluate the geometric series infinite sum* $5 + 4 + \frac{16}{5} + \frac{64}{25} + \frac{256}{125} + \cdots$.
This series is geometric with first term 5 and common ratio $\frac{4}{5}$, so we use the formula:

$$S = \frac{a}{1 - r} = \frac{5}{1 - \frac{4}{5}} = \frac{5}{\frac{1}{5}} = 25$$

5. **(7 points)** *Calculate the labeled quantities in the triangles (not drawn to scale) below.*

   (a) **(3 points)** *Determine $x$:*

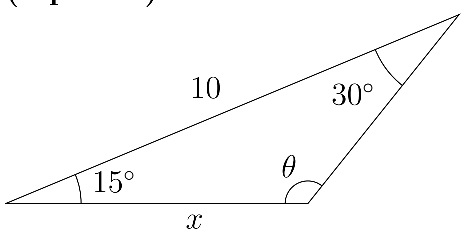



Since we have only one labeled angle but know two of the sides, the relationship among the sides is best discovered by the Rule of Cosines; we use the known angle and the two known adjacent sides to determine that:

$$\begin{aligned} x^2 &= 7^2 + 8^2 - 2 \cdot 7 \cdot 8 \cdot \cos(120°) \\ &= 49 + 64 + 108 \cdot \frac{1}{2} = 169 \\ x &= \sqrt{169} = 13 \end{aligned}$$

Note that the negative square root can be ignored, since $r$ is a length, and as such is presumably positive.

   (b) **(4 points)** *Determine $x$ and $\theta$:*




Since the angles add up to 180°, we know that $15 + 30 + \theta = 180$, so $\theta = 135°$.

In this triangle two angles and only one side have been specified, making it a prime candidate for application of the Rule of Sines. Looking at the two specified angles and their opposite sides, we get:

$$\frac{x}{\sin 30°} = \frac{10}{\sin 135°}$$

which, multiplying both sides by $\sin 30°$, gives $x = \frac{10 \sin 30°}{\sin 135°} = \frac{10\frac{1}{2}}{\frac{\sqrt{2}}{2}} = \frac{10}{\sqrt{2}}$ or $5\sqrt{2}$.

6. **(10 points)** *Answer the following questions about sequence exploration.*

   (a) **(4 points)** *The fifth term of an arithmetic sequence is 4 and the eleventh term is 40. What is the formula for the sequence?*

We set up the template for an arithmetic sequence: $a_n = a + d(n-1)$. Then $4 = a_5 = a + 4d$ and $40 = a_{11} = a + 10d$. Thus:

$$\begin{aligned} 4 &= a + 4d \\ 40 &= a + 10d \\ (40 - 4) &= (a + 10d) - (a + 4d) \\ 36 &= 6d \\ 6 &= d \end{aligned}$$

and now we can find $4 = a + 4 \cdot 6$ so $a = 4 - 24 = -20$. Thus, $a_n = -20 + 6(n-1)$.

(b) **(3 points)** *An arithmetic sequence has formula* $a_n = 5 - 3n$. *Which term in this sequence is equal to* $-34$*?*

We let $-34 = 5 - 3n$ and solve for $n$: $-39 = -3n$, so $n = 13$; thus $-34$ is the thirteenth term of the sequence.

(c) **(3 points)** *The first term of a geometric sequence is 8, and the second term is 4. What is the fifth term?*

The first term of this geometric sequence is 8; the common ratio is $\frac{4}{8} = \frac{1}{2}$, so the formula is $a_n = 8\left(\frac{1}{2}\right)^{n-1}$. Then $a_5 = 8\left(\frac{1}{2}\right)^4 = 8 \cdot \frac{1}{16} = \frac{1}{2}$.