1. **(4 points)** *Find all solutions to the equation* $4 - 2\sin(3x) = 2$.

   Note below the procedure used to invert the sine, to ensure that *all* solutions are found:

$$\begin{aligned}
4 - 2\sin(3x) &= 2 \\
-2\sin(3x) &= -2 \\
\sin(3x) &= 1 \\
3x &= \arcsin(1) + 2\pi n \text{ or } \pi - \arcsin(1) + 2\pi n \\
3x &= \frac{\pi}{2} + 2\pi n \text{ or } \pi - \frac{\pi}{2} + 2\pi n \\
3x &= \frac{\pi}{2} + 2\pi n \\
x &= \frac{\pi}{6} + \frac{2}{3}\pi n
\end{aligned}$$

   So x can take on any value of the form $\frac{\pi}{6} + \frac{2}{3}\pi n$, such as $\frac{\pi}{6}$, $\frac{5}{\pi}6$, $\frac{-\pi}{2}$, and so forth.

2. **(4 points)** *Identify each of the following sequences as arithmetic, geometric, or neither. If you identify it as arithmetic or geometric, state its common difference or ratio.*

   - $1, 2, 4, 16, 256, \ldots$.

     Note that the differences of the first 3 pairs of consecutive terms are 1, 2, and 12; these are not the same, so this sequence is not arithmetic. The ratios of the first 3 pairs of consecutive terms are 2, 2, and 4; these are not the same, so this sequence is not geometric, either.
   - $\frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, \ldots$.

     Note that the differences of the 4 pairs of consecutive terms seen here are $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{4}$; since these are the same, this sequence is arithmetic with common difference $\frac{1}{4}$.
   - $4, -12, 36, -108, 324, \ldots$.

     Note that the differences of the 4 pairs of consecutive terms seen here are -16, 48, -144, and 432; these are not the same, so this sequence is not arithmetic. The ratios of the 4 pairs of consecutive terms seen here, however, are -3, -3, -3, and 4; since these are the same, this sequence is geometric with common ratio -3.
   - $3, -1, -5, -9, -13, -17, \ldots$.

     Note that the differences of the 5 pairs of consecutive terms seen here are -4, -4, -4, -4, and -4; since these are the same, this sequence is arithmetic with common difference -4.

3. **(3 points)** Give a formula for each of the following sequences, and write the 100th term (you do not need to simplify your expression).

   - $1, 4, 9, 16, 25, \ldots$

     This sequence is recognizable as a list of small squares: $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, $25 = 5^2$. The formula for the nth term would thus be $a_n = n^2$, and the 100th term in particular would be $a_{100} = 100^2 = 10000$.
   - $\frac{1}{1}, \frac{2}{2}, \frac{3}{4}, \frac{4}{8}, \frac{5}{16}, \frac{6}{32}, \ldots$

     This sequence has recognizable things happening in both the numerator and denominator: the numerator of the nth term seems to be just n, while the denominator is a power of 2, which experimentation can show to be 2^{n-1}, so the formula is $a_n = \frac{n}{2^{n-1}}$; the 100th term would be $\frac{100}{2^{99}}$.

- $2+\sqrt{1}, 4+\sqrt{2}, 6+\sqrt{3}, 8+\sqrt{4}, 10+\sqrt{5}, \ldots$

  The terms of this series are sequential even numbers, with the square roots of sequential numbers added to them. The even numbers $2, 4, 6, 8, 10, \ldots$ can be represented in the nth term by the expression $2n$, while the terms $\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots$ have as the nth term the form $\sqrt{n}$. Thus, the nth term of the sums of these two expressions is given by $a_n = 2n + \sqrt{n}$, and for the 100th term, $a_n = 2 \cdot 100 + \sqrt{100} = 200 + 10 = 210$.

4. **(4 points)** *An arithmetic sequence has* 50 *as its sixth term and* 2 *as its tenth term. Find the sequence's first term and common difference.*

   Since we are looking at an arithmetic sequence here, we know the series is given by the formula $a_n = a + (n-1)d$ for some a and d. In particular, $a_6 = a + 5d$ and $a_{10} = a + 9d$. Taking these two known values together, and subtracting one equation from the other:

$$\begin{aligned} 50 &= a + 5d \\ 2 &= a + 9d \\ (50-2) &= (a+5d) - (a+9d) \\ 48 &= -4d \\ -12 &= d \end{aligned}$$

   so the common difference is -12. To find the first term a, all we need to do is plug this value of d back into one of our known values above:

$$\begin{aligned} 50 &= a + 5(-12) \\ 50 &= a - 60 \\ 110 &= a \end{aligned}$$

   so the first term of this sequence is 110.

5. **(3 points)** *Calculate the partial sum of the arithmetic series* $4+9+14+19+24+29+34+39+\cdots+199$.

   Taking the difference of two consecutive terms of this series, we get $9-4=5$ to be the common difference; the terms of this series are thus given by the formula $a_n = 4 + 5(n-1)$. We might start by finding out how many terms there are in this series: for what n is $a_n = 199$?

$$\begin{aligned} 199 &= 4 + 5(n-1) \\ 195 &= 5(n-1) \\ 39 &= n-1 \\ 40 &= n \end{aligned}$$

   so this sum has 40 terms. At this point we can make use of one of two approaches. If we prefer a known formula:

$$S_n = an + \frac{n(n-1)}{d}$$

   so to add up the first 40 terms, we use $n = 40$, $a = 4$, and $d = 5$ to get

$$S_{40} = 4 \cdot 40 + \frac{40 \cdot 39}{2} \cdot 5 = 160 + 3900 = 4060$$

or, alternatively, we use Gauss's termwise-summation trick:

$$
\begin{aligned}
S &= \quad 4 + \quad 9 + \ \ 14 + \ \ 19 + \ \ 24 + \ \ 29 + \cdots + 199 \\
S &= 199 + 194 + 189 + 184 + 179 + 174 + \cdots + \quad 4 \\
2S &= 203 + 203 + 203 + 203 + 203 + 203 + \cdots + 203 = 40 \cdot 203 = 8120 \\
S &= \frac{8120}{2} = 4060
\end{aligned}
$$

6. **(2 points)** *Calculate the infinite sum of the geometric series* $3 - 2 + \frac{4}{3} - \frac{8}{9} + \frac{16}{27} - \frac{32}{81} + \cdots$

Taking the ratio of two consecutive terms of this series, we get $\frac{-2}{3}$ to be the common ratio. Using the known formula for the infinite sum of a geometric series:

$$
S_\infty = \frac{a}{1-r} = \frac{3}{1-(\frac{-2}{3})} = \frac{3}{\frac{5}{3}} = \frac{9}{5}
$$

Life is an even-numbered problem.	—attribution unknown