1. **(10 points)** *Prove that if graph G is connected and contains a cycle, then there is an edge e in G such that $G - e$ is still connected.*

   Let a cycle in G be denoted by the adjacencies $v_1 \sim v_2 \sim v_3 \sim v_4 \sim \cdots \sim v_k \sim v_1$. Let e be the edge $\{v_1, v_2\}$ (in principle any edge in the cycle will work; we choose a specific one to simplify the argument). For arbitrary vertices u and v in G, connectivity of G means there is a path $u = u_1 \sim u_2 \sim u_3 \sim u_4 \sim \cdots \sim u_\ell = v$. If this path does not traverse the edge e, then u and v are connected in $G - e$ by this path; on the other hand, if the edge e occurs on this path, then we know that there is exactly one i such that $\{u_i, u_{i+1}\} = e = \{v_1, v_2\}$. There are two possibilities: either $u_i = v_1$ and $u_{i+1} = v_2$, or vice versa. These are handled nearly identically. In the first case, we consider the walk:

   $$u = u_1 \sim u_2 \sim \cdots \sim u_i = v_1 \sim v_k \sim v_{k-1} \sim \cdots \sim v_2 = u_{i+1} \sim u_{i+2} \sim \cdots \sim u_\ell = v$$

   And in the second:

   $$u = u_1 \sim u_2 \sim \cdots \sim u_i = v_2 \sim v_3 \sim \cdots \sim v_k \sim v_1 = u_{i+1} \sim u_{i+2} \sim \cdots \sim u_\ell = v$$

   In both cases we replace the traversal of the edge e in a path from u to v with a traversal of the path around the cycle the long way. Note that this is not necessarily a path; splicing together paths does not guarantee non-self-intersection, but a walk is sufficient to demonstrate connectedness. It is easy to see that e does not appear in this walk: the first and third sections consist of all edges in the path from u to v except for e, and the middle section consists of every edge in the cycle except for e. Thus, u is connected to v in $G - e$, and since u and v were arbitrary, $G - e$ is connected.

2. **(15 points)** *Recall that $\alpha(G)$, $\omega(G)$, $\delta(G)$, and $\Delta(G)$ are the independence number, clique number, minimum degree, and maximum degree of G respectively:*

   (a) **(5 points)** *Prove that $\omega(G) \leq \Delta(G) + 1$ and that $\alpha(G) \leq |G| - \delta(G)$.*

   Note that every vertex in K_n has degree $n - 1$; if $K_n \subset G$, then some vertex in G has degree of at least $n - 1$, and thus it follows that $\Delta(G) \geq n - 1$. Since $K_n \subset G$ would follow definitionally from the fact that $\omega(G) = n$, we can see that $\Delta(G) \geq \omega(G) - 1$.

   A similar proof can be shown for $\alpha(G)$, but the subgraph argument is a bit fiddlier; the easiest way to show the inequality for $\alpha(G)$ is to note that $\alpha(G) = \omega(G^c) \leq \Delta(G^c) + 1$. Since $d_{G^c}(v) + d_{G^c}(v) = |G| - 1$ for any vertex v in G, we know that $d_{G^c}(v) = |G| - d_G(v) - 1$ so $\Delta(G^c) = |G| - \delta(G) - 1$.

   (b) **(5 points)** *Give a construction mechanism for a graph G where $\delta(G)$ is arbitrarily large but $\omega(G) = 2$.*

   One easy construction is the complete bipartite graph $K_{n,n}$; $\delta(K_{n,n}) = n$, and $K_3 \not\subset K_{n,n}$, so $\omega(K_{n,n}) = 2$. Other constructions are possible.

   (c) **(5 points)** *Prove that $\omega(G) + \alpha(G) \leq |G| + 1$.*

   Let A consist of the vertices of a maximal clique in G; let B consist of the vertices of a maximal independent set in G. Thus, by definition, any two vertices in A

are adjacent, and any two vertices in B are non-adjacent. From this definition, it is self-evident that $|A \cap B| < 2$, since if $A \cap B$ contained distinct elements x and y, vertices x and y would have to be simultaneously adjacent and non-adjacent. Since A and B are both subsets of $V(G)$, it is clear that $A \cup B \subset V(G)$, and thus, $|G| = |V(G)| \geq |A \cup B| = |A| + |B| - |A \cap B| > \omega(G) + \alpha(G) - 2$. Rearranging this, we get that $\omega(G) + \alpha(G) < |G| + 2$, which, using the fact that all quantities involved are integers, can be converted to the desired inequality.

3. **(10 points)** *Prove that for any connected graph G such that $|G| > \frac{\Delta(G)^k - 1}{\Delta(G) - 1}$ and vertex u thereof, there is a vertex $v \in G$ such that $d(u, v) \geq k$.*

We shall start by proving a reasonably easy fact: for any non-negative integer r, there are no more than $\Delta(G)^r$ vertices whose distance from u is exactly r. This is provable by induction. For the base case, note that $d(u, v) = 0$ iff $u = v$, so there is exactly one point (n.b. $1 = \Delta(G)^0$) in G at a distance of zero.

For the inductive step, we observe that in order for v to be a distance r from u, there must be a shortest path of length r from u to v, which we denote $u = u_0 \sim u_1 \sim u_2 \sim \cdots \sim u_{r-1} \sim u_r = v$. Clearly $d(u, u_{r-1}) = r - 1$: we can show that it must be at most $r - 1$ by the existence of a path of length $r - 1$, and at least $r - 1$ since if $d(u, u_{r-1})$ were less than $r - 1$, then there would be a path from u to v of length less than r. Thus, every vertex at a distance r from u is adjacent to a vertex at a distance $r - 1$ from u. By our inductive hypothesis, there are at most $\Delta(G)^{r-1}$ vertices at a distance $r - 1$ from u, anr each of them have at most $\Delta(G)$ neighbors, so there can be at most $\Delta(G)^{r-1} \cdot \Delta(G) = \Delta(G)^r$ vertices at a distance r from u. (Note that in reality there are guaranteed to be fewer, and quite likely to be many fewer: a vertex at distance $r-1$ from u might not have maximum degree, or multiple such vertices might share neighbors, and at least one edge from such a vertex is guaranteed to be closer to rather than further from u. However, since we are seeking an upper bound, we can ignore these possibilities, although a slightly better upper bound would actually be $\Delta(G) \left[\Delta(G) - 1\right]^{r-1}$).

We see that with the above-proven upper bound on the number of vertices at distance of exactly r, we can also find an upper bound on the number of vertices at a distance of less than k:

$$\Delta(G)^0 + \Delta(G)^1 + \cdots + \Delta(G)^{k-1} = \frac{\Delta(G)^k - 1}{\Delta(G) - 1} < |G|$$

We thus see that not every vertex in G has distance of less than k from u, so there is some vertex v whose distance from u is k or more.

4. **(5 points)** *Let A be the adjacency matrix of graph G. Prove that G is connected if there is a value of k such that A^k has no zero entries; show additionally that the converse is not necessarily true.*

We know that the ij-th entry of A^k counts the number of walks from i to j of length k. If there is a k such that every entry is nonzero, then there is at least one walk of length k between every two points.

To show that the converse is not true, let us consider, for example, a connected bipartite graph. For simplicity, we can look at a graph as simple as K_2. Its adjacency matrix is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and it is easy to see that A^k is either A or I, depending on whether k is even or odd.

5. **(5 point bonus)** *Prove that every tree T has at least $\Delta(T)$ leaves.*

An easy way to prove this is with the structural theorem. We can prove it to be true for $|T| \leq 3$ with simple investigation of these unique small trees (we go further than usual in building our base case because the increase in $\Delta(T)$ from 1 to 2 is rather unusual.

For our inductive step, we start by noting that if $T \geq 3$ then $\Delta(T) \geq 2$, since $\|T\| = |T| - 1$, so $\frac{\sum_{v \in V(T)} d_T(v)}{|T|} = \frac{2|T|-2}{|T|} > 1$, so since the average degree is greater than 1, so is the maximum degree. Now, for a given T, the structural theorem says that T is the result of a leaf-gluing $T = T' + v + \{u, v\}$ for some tree T' and vertex $u \in T'$. Since $|T'| < |T|$, the inductive hypothesis tells us that T' has at least $\Delta(T')$ leaves. Depending on what sort of vertex u is, the addition of v can have several different effects:

Case I: $d_{T'}(u) = 1$. Then u was a leaf in T'. Note that $d_T(u) = 2$ and $d_T(v) = 1$ and for every other vertex, degrees in T are the same as in T'. Since $\Delta(T') > d_{T'}(u)$, it follows that $\Delta(T) = \Delta(T')$, since the vertices modified by the gluing procedure were not maximal and were modified by increase of at most 1. Note also that T has the same number of leaves as T': every vertex of T' except u has its leaf-status unchanged, and u is not a leaf of T, but v is, for a total change in quantity of zero.

Case II: $1 < d_{T'}(u) < \Delta(T')$. As above, since $\Delta(T') > d_{T'}(u)$, it follows that $\Delta(T) = \Delta(T')$, since the vertices modified by the gluing procedure were not maximal and were modified by increase of at most 1. However, the number of leaves increases by at least one, since all leaves of T' are also leaves of T, and so is v.

Case III: $d_{T'}(u) = \Delta(T')$. Note that in this case, since $d_T(u) = d_{T'}(u)+1 = \Delta(T)+1$, we find that $\Delta(T) = \Delta(T') + 1$; however, the number of leaves also increases by 1, using the logic displayed in Case II.

Thus, in each case, we see that adding a leaf to a tree maintains the nonstrict inequality between its maximum degree and the number of leaves in it.