This problem set is due at the beginning of class on *January 21*. Below, "graph" means "simple finite graph".

1. **(10 points)** Prove that if graph G is connected and contains a cycle, then there is an edge e in G such that $G - e$ is still connected.

2. **(15 points)** Recall that $\alpha(G)$, $\omega(G)$, $\delta(G)$, and $\Delta(G)$ are the independence number, clique number, minimum degree, and maximum degree of G respectively:

   (a) **(5 points)** Prove that $\omega(G) \leq \Delta(G) + 1$ and that $\alpha(G) \leq |G| - \delta(G)$.

   (b) **(5 points)** Give a construction mechanism for a graph G where $\delta(G)$ is arbitrarily large but $\omega(G) = 2$.

   (c) **(5 points)** Prove that $\omega(G) + \alpha(G) \leq |G| + 1$.

3. **(10 points)** Prove that for any connected graph G such that $|G| > \frac{\Delta(G)^k - 1}{\Delta(G) - 1}$ and vertex u thereof, there is a vertex $v \in G$ such that $d(u, v) \geq k$.

4. **(5 points)** Let A be the adjacency matrix of graph G. Prove that G is connected if there is a value of k such that A^k has no zero entries; show additionally that the converse is not necessarily true.

5. **(5 point bonus)** Prove that every tree T has at least $\Delta(T)$ leaves.

The origins of graph theory are humble, even frivolous. Whereas many branches of mathematics were motivated by fundamental problems of calculation, motion, and measurement, the problems which led to the development of graph theory were often little more than puzzles, designed to test the ingenuity rather than to stimulate the imagination. —*Biggs, Lloyd, and Wilson,* Graph Theory: 1736–1936