This problem set is due at the beginning of class on *February 4*. Below, "graph" means "simple finite graph".

1. **(10 points)** *Let $\tilde{K}_{n,n}$ be the bipartite graph with vertex set $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ and containing the edge $\{a_i, b_j\}$ if and only if $i \neq j$.*

   (a) **(5 points)** *Show that $\tilde{K}_{n,n}$ satisfies the Hall marriage criterion and thus has a perfect matching.*

   For $S \subseteq A$, there are three possibilities: if $|S| = 0$, then $|N(S)| = 0$, which is trivial; or $|S| = 1$, in which case $S = \{a_i\}$ and $N(S)$ consists of all b_j except b_i, so $N(S) = n - 1$; or $|S| > 1$, and then $N(S)$ consists of the entirety of B, so $N(S) = n$. The former case clearly meets the Hall criterion as long as $n \geq 2$; the latter meets it for all choices of S since $|S| \leq n = |N(S)|$.

   (b) **(5 points)** *How many different perfect matchings are there on $\tilde{K}_{n,n}$ (hint: find the number of perfect matchings on $K_{n,n}$, and exclude those which match some a_i with b_i).*

   The matchings on $K_{n,n}$ are in a one-to-one bijection with the permutations in S_n: the matching $\{\{a_1, b_{\pi(1)}\}, \{a_2, b_{\pi(2)}\}, \ldots, \{a_n, b_{\pi(n)}\}\}$ can be associated with the permuation π. The matchings in $\tilde{K}_{n,n}$ consist of those matchings on $K_{n,n}$ which do not match any a_i to b_i; this would be associated with those permutations in which no $\pi(i) = i$, which is to say, permutations without a fixed point. This is the celebrated "derangement problem", which we looked at last semester and saw via inclusion-exclusion had a solution of

$$D_n = \sum_{i=0}^{n} (-1)^n \frac{n!}{i!}$$

2. **(15 points)** *Let $f(G)$ be the number of (not necessarily perfect) matchings on a graph G.*

   (a) **(5 points)** *Show that for any edge $e = \{u, v\}$ in G, $f(G) = f(G-e) + f(G-u-v)$, where $G - e$ represents G with the edge e removed, and $G - v$ represents G with the vertex v and all incident edges removed, and that $f(G) = 1$ if $\|G\| = 0$.*

   Given an edge $e = \{u, v\}$ of G, the set of matchings on a graph G can be partitioned into two subsets: one consisting of those which use e, and one consisting of those which do not.

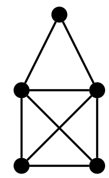
   Members of the first set can be constructed by performing a matching on all vertices of G except for the endpoints of e, and then adding e to the matching. Thus, this set is enumerated by $f(G - u - v)$.

   The second set, on the other hand, may contain edges incident on u and v, as long as those edges are not e. In order to construct a matching on G which does not contain e, we may simply perform a matching on $G - e$; there are $f(G - e)$ such matchings.

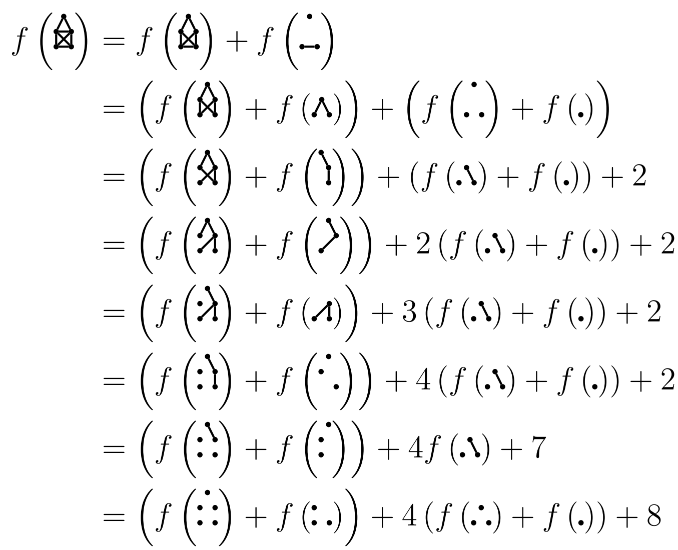
   Assembling these two parts of the overall enumeration, we see that $f(G) = f(G - e) + f(G - u - v)$.

For the base case, we observe that if there are no edges in a graph, then there is exactly one matching: the empty matching, so $f(G) = 1$ for edgeless graphs G.

(b) **(5 points)** *Using the above recurrence, find the number of matchings on the following graph:*



There is more than one way to do this, depending on choice of e at each stage of the recurrence. Choosing edges with high-degree endpoints will speed up the process, however.



$= 18$

(c) **(5 points)** *How could the above recurrence be modified to give the number of* perfect *matchings?*

Note that the partition approach used in part (a) would remain valid even if "matching" were replaced by "perfect matching" throughout, so the nubmer of perfect matchings $g(G)$ is also subject to the recurrence $g(G) = g(G-u-v)+g(G-e)$. However, the last section of part (a), in whch we explore initial conditions, is *not* true for perfect matchings. A graph with vertices but no edges has zero perfect matchings, while a graph without vertices at all has exactly one. Thus, we would need to change our initial conditions to be that $g(G) = 1$ for $|G| = 0$, and $g(G) = 0$ for $\|G\| = 0$ and $|G| > 1$.

3. **(10 points)** *Prove Hall's Theorem by restricting Tutte's Theorem to the bipartite case and exhibiting that the Hall criterion follows from the Tutte criterion if G is bipartite.*

It's somewhat easier to prove the converse of the above statement: the existence of a set S not satisfying the Hall criterion $|S| \leq |N(S)|$ implies the existence of a set T not satisfying the Tutte criterion, so that $G - T$ has more than T odd components.

So, if G does not meet the conditions of Hall's Theorem, then there is a vertex-set $S \subset A$ such that $|S| > |N(S)|$. Let $T = N(S)$. By the definition of a neighborhood, every element of S is adjacent to an element of T, so every element of S is an isolated vertex of $G - T$. Thus, $G - T$ contains at least $|S|$ isolated vertices, which are in themselves odd components; since $|S| > |T|$, this set violates Tutte's Theorem.

4. **(5 points)** *Show without using Menger's Theorem that if G is 2-connected and u and v are distinct vertices of G, there is a cycle in G containing both u and v.*

There are several approaches to this one, but the least troublesome is to use the structural theorem, which says that every 2-connected graph is either a cycle or a path grafted onto a smaller 2-connected graph. Our argument is essentially inductive on this process (formally, it would be an induction on the number of steps within the structural construction of the graph G).

We will start by proving the subordinate result that, for any three vertices u, v, and w in a 2-connected graph, there is a path from u through v to w. This is trivially true on a cycle; if we assume it is true on a graph $G = G' + P$, then it

As a base case, we consider the basic 2-connected graphs: the cycles. Trivially, any two points on a cycle lie on the same cycle.

Now, for our inductive step, we assume the above statement to be true for some 2-connected G', and then consider $G = G' + P$, where P is a path between some vertices u and v of G'. For arbitrary x and y in G, we want to show x and y lie on a cycle. This is quite easy by explicit consideration of the several possibilities: if $x, y \in G'$, then the inductive hypothesis tells us there is a cycle containing x and y in G', which also lies in G. If $x, y \in P - \{u, v\}$, then by connectivity of G', there is a path from u to v not through P, and this path together with P forms a cycle. The only complicated case is when $x \in G'$ and $y \in P - \{u, v\}$.

This case is easy as long as we can get independent paths from x to u and v in G': then we can merge those paths, since they're independent, to get a path from u to v via x, and then merge this path with P to get a cycle through x and y. So, we shall see how to get these independent paths in G'.

Using the inductive hypothesis, there is a cycle containing x and u; let us decompose it into two independent paths P_1 and P_2 from x to u. In addition, by 2-connectivity of G', there is a path from x to v in $G' - u$, which we shall denote $x = v_0 \sim v_1 \sim \cdots \sim v_n = v$. We know that no $v_i = u$ by the construction, however, it is possible that this path intersects P_1 and P_2. If it does not intersect either of them, except at $x = v_0$, then the independent paths sought are this path together with either P_1 or P_2.Otherwise, let i be the largest index such that $v_i \in P_1 \cup P_2$; since $v_i \neq x$, it lies in exactly one of P_1 or P_2. Without loss of generality, assume $v_i \in P_2$. Then, we have two independent paths constructed as follows: P_1 is a path from x to u, and a path can be constructed from the fragment of P_2 from x to v_i spliced with the path $v_i \sim v_{i+1} \sim \cdots v_n = v$.