1. **(15 points)** *Demonstrate the following facts about a directed graph D.*

   (a) **(5 points)** *Prove that $\sum_{v\in V(D)} d_D^-(v) = \sum_{v\in V(D)} d_D^+(v)$. Recall that d^- and d^+ represent the indegree and outdegree respectively.*

   Note that $d^-(v) = |\{(v,u) \in E(G)\}| = \sum_{u\leftarrow v} 1$, and similarly with opposite directionality for $d^+(v)$. Thus:

$$\sum_{v\in V(D)} d_D^-(v) = \sum_{v\in V(D)} \sum_{u\leftarrow v} 1 = \sum_{\substack{u,v\in V(G)\\(v,u)\in E(G)}} 1 = |E(G)|$$

   Likewise, $\sum_{v\in V(D)} d_D^+(v) = |E(G)|$.

   (b) **(10 points)** *Prove that a directed Eulerian tour (i.e. a directed closed trail traversing every edge) on a strictly connected graph D exists if and only if $d^-(v) = d^+(v)$ for all $v \in V(D)$.*

   Let us start by supposing D has an eulerian tour: we shall show that all vertices of D have equal indegree and outdegree. Let us denote our eulerian tour by $v_1 \to v_2 \to v_3 \to v_4 \to \cdots \to v_m \to v_1$, and let $e_i = (v_i, v_{i+1})$, with the special case $e_m = (v_m, v_1)$. Then, by the definition of an eulerian tour, all the e_i are distinct and $E(G) = \{e_1, e_2, \ldots, e_m\}$. Now, consider a vertex u in G. Some number k of the v_i are equal to u: let us denote $u = v_{i_1} = v_{i_2} = \cdots = v_{i_k}$. Then, for any j, note that $v_{i_j\pm 1}$ (using the wraparound convention that $v_0 = v_k$ and $v_{k+1} = v_1$ where applicable) must *not* be identical to u, since $v_{i_j\pm 1}$ is adjacent to u and G is a simple digraph. Thus, since none of the i_j are consecutive, the indices $\{i_1 - 1, i_i, i_2 - 1, i_2, \ldots, i_k - 1, i_k\}$ are all distinct. Since u is the head of e_i if and only if either $i = i_j$ and u is the tail if and only if $i = i_j - 1$, it follows that the above set of indices is exactly the indices of the edges incident on u, and exactly half of them are incoming, and half of them outgoing. Since every element of $E(G)$ is associated with exactly one index, the set of edges incident on u is equinumerous with this index set, so $d^+(u) = d^-(u) = k$.

   To prove that equal indegree and outdegree at every vertex is sufficient, we dispense with the trivial case where D consists only of an isolated vertex and prove the nontrivial digraphs are eulerian by contradiction. Consider a nontrivial connected digraph D in which every vertex has equal indegree and outdegree. Since D is strictly connected and nontrivial, $\delta^+(D) \geq 1$ and $\delta^-(D) \geq 1$, since there must be paths to and from every vertex in D. As seen when investigating acyclic graphs, this property guarantees that D contains a directed cycle. Let C be a maximal directed tour in D. If C is not an eulerian tour, then there is at least one edge e in $D - C$. Either e is incident on a vertex of C, or there is a vertex in v not in C, and thus there is a path from some vertex of C to v; in either case we are guaranteed that some vertex u in C is incident on an edge in $D - C$. Let H be the strong component of $D - C$ containing u. Since u is incident on an edge, H is nontrivial and connected; furthermore, since a tour, as seen in the necessity argument, utilizes the same number of incoming and outgoing edges from each vertex, the degrees in H have the same equality of incoming-and-outgoing edges property seen in D. Thus, $\delta^+(H) = \delta^-(H) = 1$, so H contains some cycle C'. Since C' and C share no edges and share at least one vertex u, they can be spliced together to get a tour whose length is the sum of the lengths of C and C', contradicting C's maximality.

Note: this proof was based very closely (to the extent of being a modification of a copy and paste) on the proof that even degree was necessary and sufficient for an undirected graph tobe eulerian.

2. **(10 points)** *Suppose each edge in a digraph D with a source s and sink t has not only a maximum capacity $c(e)$ but also a* minimum required flow *$b(e)$, so that a flow f must satisfy $b(e) \leq f(e) \leq c(e)$. Assume we know of a legal but not necessarily maximal flow f_0. Explain how the Ford-Fulkerson algorithm could be modified to produce a maximum flow under these constraints.*

   There are only a few modifications required. First, we can no longer start with the zero flow, which is quite likely not legal, but instead start with f_0 and improve it. Our other change is a modification of the definition of "improvability" on backwards edges. "Improvability" in this context means the ability to be reduced (rather than increased), which on the original problem was true for any positive flow, but now is true if $f(e) > b(e)$. Similarly, the quantity by which it is considered to be improvable (denoted q_i in our original problem) is, instead of $f(e)$, now $f(e) - b(e)$. In total, the resulting algorithm is as below, with modifications underlined.

   (a) Let $f = \underline{f_0}$.

   (b) Let $s \in S$.

   (c) For any $x \in S$ from which we have not yet "probed", consider each neighbor $y \notin S$ of x. If either $x \to y$ with $f(x, y) < c(x, y)$ or if $y \to x$ with $f(y, x) > \underline{b(y, x)}$, then put y in S, recording that its "parent" is x.

   (d) If $t \in S$, then skip to step 7.

   (e) If there are still unprobed vertices in S, go back to step 3.

   (f) There are no unprobed vertices in S and $t \notin S$, so f is a maximal flow!

   (g) Identify t as x_0, t's parent as x_1, the parent of x_1 as x_2, and so forth, until reaching $x_k = s$.

   (h) For each i, let $q_i = c(x_{i+1}, x_i) - f(x_{i+1}, x_i)$ or $f(x_i, x_{i+1}) \underline{- b(x_i, x_{i+1})}$ as appropriate. Let $q = \min q_i$.

   (i) For each i, either increment $f(x_{i+1}, x_i)$ by q or decrement $f(x_i, x_{i+1})$ by q, as appropriate.

   (j) With this improved flow f defined, forget which elements are in S and return to step 2.

   Note that if $b(e) = 0$ for every edge e, and f_0 is the zero flow, this is identical to the traditional Ford-Fulkerson algorithm.

3. **(10 points)** *Let us consider a problem of* committee assignment *akin to traditional matching problems. Suppose we have individuals* $x_1, x_2, \ldots, x_k$ *and committees* $c_1, c_2, \ldots, c_\ell$, *such that each individual* x_i *has qualifications to serve on some subset* S_i *of the committees, and in addition, each committee* c_j *can only have at most* n_j *members. The goal is to assign as many individuals to committees as possible.*

   (a) **(5 points)** *Produce a bipartite graph on* $k + \sum_{j=1}^{\ell} n_j$ *vertices on which the committee-assignment problem may be solved by finding a maximal matching.*

   We can unfold the committee c_j into n_j identical "chairs" $y_{j,1}, y_{j,2}, \ldots, y_{j,n_j}$, to each of which we assign a vertex. Now, if x_i is qualified to serve on a committee c_j, we allow it to be plausibly matched to each of the "chais" in committee c_j, or, in other words, put the edge $x_i, c_{j,k}$ into our graph for each k. A maximal matching on this graph would assign each individual to a "chair", and then, ingoring the enumeration of the chairs within a committee, we would have an assignment of as many individuals as possible to committees.

   (b) **(5 points)** *Produce a digraph on* $k + \ell + 2$ *vertices on which the committee-assignment problem may be solved by finding a maximal integer flow.*

   Let us concoct a system in which each individual assignment from x_i to c_j corresponds to a single unit of flow $s \to x_i \to c_j \to t$. Then, we will have edges from s to each x_i of capacity 1, since each individual can be assigned exactly once. Then, we will have edges of capacity 1 (or greater; it doesn't actually matter) from each x_i to committees c_j in S_i, since we want to only permit flows from individuals to committees for which they are qualified. And finally, since a committee c_j could have as many as n_j members contributing inflows, we want the edges from c_j to t to have capacity n_j. Now, since committee assignments are identified specifically with units of flow, a maximum assignment is identical to a maximum flow.

4. **(5 points)** *Prove that if* D *is a digraph on* n *vertices such that no two vertices have the same indegree, then* D *is acyclic.*

   Let us proceed by induction on n; the $n-1$ case is trivial. Since there are only n possible indegrees (integers from 0 to $n-1$) each must be possessed by exactly one vertex. In particular, there is a vertex v of indegree $n-1$, which is adjacent via incoming edges to every vertex. Since all edges to this vertex are incoming, $D - v$ has the same indegrees on all vertices as D itself does, so $D - v$ consists of $n-1$ vertices with distinct indegrees, so by the induction hypothesis, $D - v$ is acyclic; thus, any cycle in D must contain v. However, since all edges to v are incoming, it cannot appear on any cycles.

5. **(5 point bonus)** *Prove that there is a one-to-one correspondence between the posets on a set of* n *elements and the transitive digraphs on* n *labeled vertices.*

   Let a transitive digraph D induce a relation $(S, \preceq)$ in the following manner: $S = V(D)$, and $u \preceq v$ if either $u = v$ or $u \to v$. We shall see that $\preceq$ is a partial ordering: it is reflexive by definition, antisymmetric since if $u \to v$, $v \not\to u$, and transitive by translation of the graph transitivity condition.

Likewise, if we have a poset $(S, \preceq)$, we may build a digraph D by letting $V(D) = S$, and such that $u \to v$ if $u \neq v$ and $u \preceq v$. Then, D will be transitive as a consequence of the transitivity condition of posets.