This problem set is due at the beginning of class on *February 25*. Below, "graph" means "simple finite graph" except where otherwise noted.

1. **(15 points)** Demonstrate the following facts about a directed graph D.

   (a) **(5 points)** Prove that $sum_{v\in V(D)} d_D^-(v) = \sum_{v\in V(D)} d_D^+(v)$. Recall that d^- and d^+ represent the indegree and outdegree respectively.

   (b) **(10 points)** Prove that a directed Eulerian tour (i.e. a directed closed trail traversing every edge) exists if and only if $d^-(v) = d^+(v)$ for all $v \in V(D)$.

2. **(10 points)** Suppose each edge in a digraph D with a source s and sink t has not only a maximum capacity $c(e)$ but also a *minimum required flow* $b(e)$, so that a flow f must satisfy $b(e) \leq f(e) \leq c(e)$. Assume we know of a legal but not necessarily maximal flow f_0. Explain how the Ford-Fulkerson algorithm could be modified to produce a maximum flow under these constraints.

3. **(10 points)** Let us consider a problem of *committee assignment* akin to traditional matching problems. Suppose we have individuals $x_1, x_2, \ldots, x_k$ and committees $c_1, c_2, \ldots, c_\ell$, such that each individual x_i has qualifications to serve on some subset S_i of the committees, and in addition, each committee c_j can only have at most n_j members. The goal is to assign as many individuals to committees as possible.

   (a) **(5 points)** Produce a bipartite graph on $k + \sum_{j=1}^{\ell} n_j$ vertices on which the committee-assignment problem may be solved by finding a maximal matching.

   (b) **(5 points)** Produce a digraph on $k + \ell + 2$ vertices on which the committee-assignment problem may be solved by finding a maximal integer flow.

4. **(5 points)** Prove that if D is a digraph on n vertices such that no two vertices have the same indegree, then D is acyclic.

5. **(5 point bonus)** Prove that there is a one-to-one correspondence between the posets on a set of n elements and the transitive digraphs on n labeled vertices.

Life is like a sewer. What you get out of it depends on what you put into it.
—Tom Lehrer