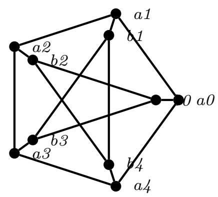
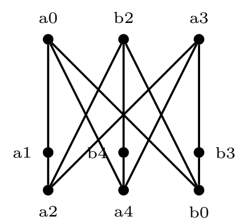
This problem set is due at the beginning of class on *March 23*. Below, "graph" means "simple finite graph" except where otherwise noted.

1. **(10 points)** *The Petersen graph is shown below.*



(a) **(5 points)** *Demonstrate that the Petersen graph is nonplanar by invoking Kuratowski's Theorem.*

The above graph has been labeled with vertices $a_0, \dots, a_4, b_0, \dots, b_4$ where each $a_i \sim a_{i\pm 1}$ and $b_i \sim b_{i\pm 2}$. Let us rearrange this graph slightly, putting a_0, b_2, and a_3 on the top, and b_0, a_2, and a_4 on the bottom. This choice of 6 points was made to maximize the adjacencies between the top and bottom sets; specifically, we chose vertices lying on a cycle of size 6. Drawn this way, we see that this will specifically have the subgraph



which is a subdivision of $K_{3,3}$.

(b) **(5 points)** *Demonstrate that the Petersen graph is nonplanar by invoking Euler's theorem.*

First, let's note that the Petersen graph contains no cycles of length 3 or 4. Let $a_0, \dots, a_4, b_0, \dots, b_4$ be as above. Since every a_i is adjacent to one and only one b_i and vice versa, it is clear that the Petersen graph contains no C_3, since if it were among the vertices $\{a_i, a_j, b_k\}$, both a_i and a_j would need to be adjacent to b_k; likewise for $\{a_i, b_j, b_k\}$. The triple $\{a_i, a_j, a_k\}$ also cannot be a C_3 since the only cycle among only the a_i vertices is the outer C_5, and likewise for $\{b_i, b_j, b_k\}$.

Showing no cycle of length 4 exists is only slightly more tedious. The possibility of a cycle with 4 a vertices of 4 b vertices is dispensed with above. Likewise, 3 a vertices and 1 b vertex is impossible, since b_i would need to be adjacent to distinct a_j and a_k. A slight modification of this argument serves to eliminate the possibility of 3 b vertices and one a. Thus, there must be 2 a vertices and 2 b vertices. If they are in the order $a_i \sim b_j \sim a_k \sim b_\ell \sim a_i$, then we have the same problem as noted in the 3–1 split case; if, on the other hand, we have $a_i \sim a_j \sim b_k \sim b_\ell \sim a_i$, then it must be the case that $j = k$ and $i = \ell$, and likewise

$k \equiv \ell \pm 2 \bmod 5$ while $j \equiv i \pm 1 \bmod 5$, yielding $\pm 2 \equiv \pm 1 \bmod 5$, which is false. Thus, no cycle of length 4 lies in the Petersen graph.

The Petersen graph has 15 edges and 10 vertices; if it were planar than by Euler's Theorem it would have $15 - 10 + 2 = 7$ faces. Since each face has a cycle on its boundary, so it is bounded by at least 5 edges, and since each edge can appear in at most 2 faces, in order to have 7 faces the Petersen graph would need at least $\frac{7 \cdot 5}{2} = 17.5$ edges, which it does not have.

2. **(10 points)** *Prove the following results about chromatic number:*

   (a) **(5 points)** *Show that on any graph G, $\chi(G) \geq \frac{n}{\alpha(G)}$.*

   Suppose G is k-colorable, so there is a proper coloring $c : V(G) \to \{1, 2, \ldots, k\}$. Then, we may partition the vertices of G according to their color, defining $V_1, \ldots, V_k$ as the sets so that $v \in V_i$ iff $c(v) = i$. By the propriety of the coloring, we know that if $u, v \in V_i$, then $u \not\sim v$, since two vertices of the same color cannot be adjacent. Thus, each V_i is an independent set, since it consists of mutually nonadjacent vertices, and thus each $|V_i| \leq \alpha(G)$, so

   $$|G| = |V_1| + \cdots + |V_k| \leq k\alpha(G)$$

   Since we know that G is k-colorable for $k = \chi(G)$, it follows that $\frac{|G|}{\alpha(G)} \leq \chi(G)$.

   (b) **(5 points)** *Show that if G has decomposition into blocks $B_1, B_2, \ldots, B_n$, then $\chi(G) = \max_i(\chi(B_i))$.*

   An easy way to prove this is by appeal to the known block diagram, which is a tree, and use induction on n. In the case of $n = 1$, $G = B_1$, so $\chi(G) = \chi(B_1)$.

   Now, given G decomposed into blocks $B_1, \ldots, B_n$, the block diagram is a tree, and thus at least one block is a leaf of the block diagram. Without loss of generality we may enumerate our blocks such that B_n is a leaf. We will then know from the construction of the block diagram that B_n intersects exactly one other block, and in exactly one vertex. Let G' be the graph with blocks $B_1, \ldots, B_{n-1}$. By the inductive hypothesis, $\chi(G') = \max_{i \leq n-1}(\chi(B_i)$. Let us define $k' = \chi(G')$, and let $k = \chi(B_n)$. Thus, G' is k'=colorable, and G is k-colorable, so both G' and G are $\max(k, k')$-colorable, and we may denote such k-colorings by $c_{G'}$ and c_{B_n}. Let the vertex where B_n intersects G' be denoted v. If $c_{G'}(v) = c_{B_n}(v)$, the two colorings can be trivially spliced together to form a coloring on G; otherwise, a permutation will be necessary. Let π be any permutation on $\{1, 2, \ldots, \max(k, k')\}$ such that $\pi(c_{B_n}(v)) = c_{G'}(v)$. Since swapping the values of colors has no effect on the integrity of the coloring, $\pi(c_{B_n})$ is a proper coloring just like c_{B_n} is, and it can be spliced with $c_{G'}$ to get a coloring on G. Thus G is $\max(k, k')$-colorable. Clearly G is not $(\max(k, k') - 1)$-colorable, since such a coloring restricted to G' or B_n would either give a $(k' - 1)$-coloring of G' or a $(k - 1)$-coloring of B_n, which cannot exist. Thus, $\chi(G) = \max(k, k') = \max_i(\chi(B_i)$.

3. **(15 points)** *A greedy coloring of a graph with an ordered set of vertices $v_1, v_2, \ldots, v_n$ is produced by labeling each vertex in order with the lowest number not already used at an adjacent labeled vertex.*

(a) **(5 points)** *Show that, for any graph G, there is some ordering of the vertices on which the greedy coloring uses only $\chi(G)$ colors.*

For brevity, we shall denote $k = \chi(G)$. G definitionally has a k-coloring $c : V(G) \to \{1, 2, \ldots, k\}$. Let $V_1, V_2, \ldots, V_k$ be the partition of $V(G)$ into color classes; that is, $v \in V_i$ if $c(v) = i$. Let us impose an aribtrary ordering $(v_{i,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,|V_i|})$ on each V_i, and assemble an overall ordering of the vertices of G:

$$(v_{1,1}, v_{1,2}, \ldots, v_{1,|V_1|}, v_{2,1}, v_{2,2}, \ldots, v_{2,|V_2|}, \ldots, v_{k,1}, v_{k,2}, \ldots, v_{k,V_k})$$

Let c_g be the greedy cloring induced by this ordering. We shall see that $c_g(v) \leq c(v)$ for all v, and thus c_g uses only k colors (it cannot use any fewer, since G is not $(k-1)$-colorable).

We will argue that $c_g(v_{i,j}) \leq c(v_{i,j})$ via induction on i. For $i = 1$, each $v_{1,j}$ has no colored neighbors (since the $v_{1,j}$ vertices are colored first, and are not adjacent to each other), so the greedy coloring assigns them the color 1, so $c_g(v_{1,j}) = 1 = c(v_{1,j})$.

For the inductive step, let us consider the step at which the greedy coloring chooses a color for $v_{i,j}$. At this point, the greedy coloring has assigned colors only to vertices of the form $v_{i',j'}$ for $i' < i$, or $v_{i,j'}$, where $j' < j$. Vertices of the latter form are in V_i, and must be nonadjacent to $v_{i,j}$. Thus, the neighbors of $v_{i,j}$ which have already been colored are all of the form $v_{i',j'}$ for $i' < i$. By the inductive hypothesis, each $c_g(v_{i',j'}) \leq c(v_{i',j'}) = i' < i$, so every neighbor of $v_{i,j}$ which has been colored has been assigned a color less than i. Thus, since $v_{i,j}$ is assigned the smallest color not used in its neighborhood, it is assigned a color less than or equal to i, so $c_g(v_{i,j}) \leq i = c(v_{i,j})$.

Note that the greedy algorithm is not guaranteed to find the particular $\chi(G)$-coloring c, and in fact there are minimal colorings not achievable by greedy coloring; for instance, if we have a K_3 with vertices $\{v_1, v_2, v_3\}$, and we attach a vertex u adjacent to v_3, and assign the coloring $c(v_1) = 1$, $c(v_2) = 2$, $c(v_3) = 3$, and $c(u) = 2$, no greedy coloring will ever match this coloring, since u will preferentially be assigned color 1 by a greedy algorithm.

(b) **(5 points)** *Demonstrate that for any $k > 2$ there is a bipartite graph G on $2k$ vertices and vertex ordering thereon in which the greedy coloring uses k colors, even though $\chi(G) = 2$.*

Let G have parts $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$, such that $a_i \sim b_j$ iff $i \neq j$. Now let us investigate the results of a greedy coloring with the vertex order $\{a_1, b_1, a_2, b_2, \ldots, a_k, b_k\}$. Each of a_1 and b_1 will be assigned color 1, since they are not adjacent to any already-colored vertices. The vertices a_2 and b_2, which are each adjacent to a vertex already assigned color 1, will be assigned color 2. Then a_3 and b_3, each adjacent to vertices of colors 1 and 2, must be assigned color 3. This procedure can be continued indefinitely, until a_k and b_k, each adjacent to vertices of every color from 1 through $k-1$, must be assigned color k.

(c) **(5 points)** *Show that a greedy coloring of the cycle C_n, regardless of the value of n, does not use more than 3 colors.*

Suppose some vertex ordering of C_n causes a vertex v to be color 4 in the greedy coloring. Thus, v must have neighbors assigned colors 1, 2, and 3. But v only has 2 neighbors in total, so this is impossible.

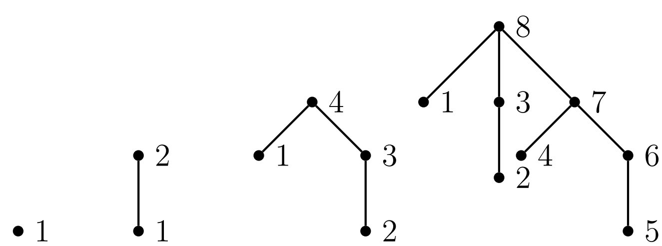
Note that there are vertex orderings which assign 3 colors to even cycles: for instance, if you place the vertices of C_6, using the canonical cyclic vertex labels, in the order $(v_1, v_4, v_2, v_3, v_5, v_6)$, then v_3 and v_6 will both be assigned color 3.

4. **(5 points)** *Prove that a bipartite graph on $n > 2$ vertices is planar only if it has $2n-4$ or fewer edges.*

   First, note that if a bipartite graph G is a tree, then it has $n-1$ edges and trivially satisfies the above criterion, so we shall consider only graphs which contain cycles. A bipartite graph can contain no odd cycles and specifically no C_3s, so every cycle has length 4 or larger. Suppose G is a planar bipartite non-tree. Since G contains a cycle, every face in a planar projection of G has at least one cycle on its boundary, so the boundary of any face consists of at least 4 edges. Since each edge appears in at most 2 faces, we shall see that G has no more than $\frac{2\|G\|}{4} = \frac{\|G\|}{2}$ faces. Using Euler's Theorem, we then see that $f(G) + n = \|G\| + 2$. Using the above-discovered inequality on the number of faces, it thus follows that $\frac{\|G\|}{2} + n \geq \|G\| + 2$, from which it can be seen that $\|G\| \leq 2(n-2)$.

5. **(5 point bonus)** *Prove that for every value of $k > 2$, there is a tree and ordering of the vertices thereon such that a greedy coloring (see above) of the tree requires k colors. What is the smallest such tree you can find?*

   Let the ordered tree T_i be defined recursively: $T_1 = \{v_1\}$, with the unique ordering on a single vertex. Then, $T_i = T_1 \cup T_2 \cup \cdots \cup T_{i-1} \cup \{v_i\}$ with the ordering $(T_1, T_2, \ldots, T_{i-1}, v_i)$, and with v_i adjacent to the last vertex from each of the T_j for $j < i$. An example of the first few steps of this construction is given below:



   There is a simple inductive argument to show why this construction will work: each T_i has a head vertex in color i, so if we add a vertex v_{i+1} adjacent to head vertices from each T_i and color it last, it must be assigned color $i+1$. We may also inductively show that the tree T_i will have 2^{i-1} vertices. To the best of my knowledge, this is the smallest such construct, but I have no proof of it.