1. **(10 points)** *Complete the proof that the Harary graphs are k-connected. You may use the case presented in class of even values of k, either by citation or imitation.*

   (a) **(5 points)** *Show that $H_{n,k}$ is k-connected for even n and odd k. This graph is symmetric, so you may specifically demonstrate connectedness between v_1 and an arbitrary v_i in $H_{n,k} - S$ for $v_1, v_i \notin S$ and $|S| < k$.*

   Let $k = 2r + 1$, and consider $|S| < k$. The exact technique described in class can be used to "crawl over" any cluster of vertices in $H_{n,k} - S$ in which S does not have r consecutive elements. Thus, either $\{v_2, \ldots, v_{i-1}\}$ is navigable this way, or $\{v_{i+1}, \ldots, v_n\}$ is, or S consists specifically of r consecutive elements of $\{v_2, \ldots, v_{i-1}\}$ and r consecutive elements of $\{v_{i+1}, \ldots, v_n\}$.

   Since only one configuration of S presents difficulties not addressed in a previous proof, let us consider specifically the case where $S = \{v_2, v_3, \ldots, v_{r+1}, v_a, v_{a+1}, \ldots, v_{a+r}\}$, for $r + 1 < b < b + r \leq n$. It is obvious by our limitations on the selection of S that this graph has at most two components: $\{v_{a+r+1}, v_{b+r+2}, \ldots, v_n, v_1\}$ and $\{v_{r+2}, v_{r+3}, \ldots, v_{a-1}\}$; any diameter between these two sets will imply that they are a single component and that $H_{n,k} - S$ is connected. Since $r < \frac{k}{2} \leq \frac{n}{2}$, the diameter $\{v_1, v_{1+\frac{n}{2}}$ is such an edge unless either $a + r < 1 + \frac{n}{2}$ or $a \leq 1 + \frac{n}{2} \leq a + r$. In the former case, the entirety of S lies in $v_1, \ldots, v_{\frac{n}{2}}$, so every point in that range ont in S is still incident on a diameter, guaranteeing connectedness; in the latter case, we see that the diameter $\{v_{a-1}, v_{a-1+\frac{n}{2}}\}$ must exist, since $a + r < a - 1 + \frac{n}{2} \leq n$.

   (b) **(5 points)** *Show that $H_{n,k}$ is k-connected for odd n and odd k. This graph is not symmetric, so you must distinguish between the cases where $v_1 \in S$ and $v_1 \notin S$.*

   As above, let $k = 2r + 1$, and consider $|S| < k$, and since $H_{n,k-1} \subset H_{n,k}$, the same argument as seen in class and invoked above is true except when S consists of two strings of r consecutive vertices.

   Unfortunately, every vertex of $H_{n,k}$ is not identical in this case; while before we assumed one of these strings started at v_2, here we must consider more possibilities. Let us consider $S = \{v_a, v_{a+1}, \ldots, v_{a+r}, v_b, v_{b+1}, \ldots, v_{b+r}\}$, where we consider our addition modulo n to concisely state cyclicity results. It is obvious by our limitations on the selection of S that this graph has at most two components: $\{v_{b+r+1}, v_{b+r+2}, \ldots, v_{a-1}\}$ and $\{v_{a+r+1}, v_{a+r+2}, \ldots, v_{b-1}\}$; any quasi-diameter between these two sets will imply that they are a single component and that $H_{n,k} - S$ is connected. Since $r < \frac{k}{2} \leq \frac{n-1}{2}$ vertices are in S, we know that there is some pair of vertices adjacent via a quasidiameter neither of which lies in S (we can partition $V(H_{n,k})$ into $(n-1)$ adjacency classes and a leftover vertex; not every adjacency class can be represented in S). If this quasidiameter connects the two sets mentioned above, we're done; otherwise the entirety of S lies in a single half-circle of $H_{n,k}$, so any vertices lying in that half-circle have their quasidiameter-neighbors lying outside of S, so any vertices between the two parts of S lie on an intact quasidiameter.

2. **(15 points)** *Answer the following questions about Turán numbers and extremal graphs.*

   (a) **(5 points)** *Show that the extremal number $\mathrm{ex}(n, K_{1,r})$ is $\frac{(r-1)n}{2}$ if r is odd or n is even, and $\frac{(r-1)n-1}{2}$ if r is even and n is odd.*

   Clearly, G contains a $K_{1,r}$ subgraph if and only if $\Delta(G) \geq r$: if $d(v) \geq r$, then v and r of its neighbors are a $K_{1,r}$, and conversely, if v and r of its neighbors form a $K_{1,r}$ subgraph, then v must have at elast r neighbors so $\Delta(G) \geq r$.

Thus, G is $K_{1,r}$-free if and only if $\Delta(G) < r$, or, in other words, $\Delta(G) \leq r-1$. Thus we know that, if $|G| = n$,

$$\|G\| = \frac{1}{2}\sum_{v\in V(G)} d(v) \leq \frac{1}{2}\sum_{v\in V(G)} \Delta(G) \leq \frac{n(r-1)}{2}$$

Since $\|G\|$ must be an integer, this bound on the maximum number of edges in G can be lowered to $\frac{n(r-1)-1}{2}$ when n is odd and $r-1$ is even, so this bound is identical to the asserted value of $\text{ex}(n, K_{1,r})$ in the problem; however, to show that this is the value of $\text{ex}(n, K_{1,r})$ and not just its upper bound, we must show that it can be achieved. The Harary graph $H_{n,r-1}$ will suffice for this purpose.

(b) **(5 points)** *Find a formula for* $\text{ex}(n, P_4)$ *by finding necessary conditions for a graph to not contain a* P_4 *(i.e. a path with 4 vertices and 3 edges). Exploration on small values of* n *may be helpful.*

There are several structures we see can not be in a P_4-free graph. Obviously, any cycle of size 4 or more is forbidden; likewise, a cycle of size 3 with any adjacent edge cannot be in it. Thus, we already have limted the possible P_4-free graphs down to those in which every component either consists of an isolated 3-cycle, or is cycle-free (and thus a tree). Since a 3-cycle has as many edges as vertices, and a tree has one fewer edge than vertices, we see that $\text{ex}(n, P_4) \leq n$, with equality attainable only when n is a multiple of 3 (since only in that case can all vertices be evenly distributed among isolated 3-cycles, with no components which are trees).

In fact, this bound of $n-1$ edges when n is nondivisible by 3, and n edges when n is divisible by 3, can be shown to be achievable: the star $K_{1,n-1}$ has $n-1$ edges and is P_4-free. In the particular case when n is divisible by 3, dividing the vertices evenly between C_3 components allows us to achieve n edges. Thus:

$$\text{ex}(n, P_4) = \begin{cases} n \text{ if } 3 \mid n \\ n-1 \text{ if } 3 \nmid n \end{cases}$$

(c) **(5 points)** *Prove that for nontrivial* H*, the asymptotic density of any Turán graph has the bounds:* $\frac{\chi(H)-2}{\chi(H)-1} \leq \lim_{n\to\infty} \frac{\text{ex}(n,H)}{\binom{n}{2}} \leq \frac{|H|-2}{|H|-1}$

We know that $\lim_{n\to\infty} \frac{t_{n,r}}{\binom{n}{2}} = \frac{r-1}{r}$, where $t_{n,r}$ is the number of edges in the n-vertex r-partite Turán graph, so all we really need to do here is show that $t_{n,\chi(H)-1} \leq \text{ex}(n, H) \leq t_{n,|H|-1}$. Both of these actually end up being reasonably easy: since H is $\chi(H)$-colorable, it is $\chi(H)$-partite. The Turán graph $T_{n,\chi(H)-1}$ is $(\chi(H)-1)$-partite, and thus must be H-free, since it has too few parts to contain a copy of H. Since $T_{n,\chi(H)-1}$ is H-free, it follows that $\text{ex}(n, H) > |T_{n,\chi(H)-1} = t_{n,\chi(H)-1}$. To demonstrate the other bound, note that $H \subset K_{|H|}$, so any H-free graph is also $K_{|H|}$-free; specifically, an extremal H-free graph of size $\text{ex}(n, H)$ is $K_{|H|}$-free, so $\text{ex}(n, K_{|H|}) \geq \text{ex}(n, H)$. But we know $\text{ex}(n, K_{|H|})$ to be equal to $t_{n,|H|-1}$, by Turán's Theorem.

3. **(10 points)** *Answer the following questions about Ramsey numbers.*

(a) **(5 points)** *Show that the* r*-color Ramsey number* $R(k_1, k_2, \ldots, k_r)$ *is subject to the recurrence inequality:*

$$R(k_1, k_2, \ldots, k_r) \leq R(k_1-1, k_2, \ldots, k_r) + R(k_1, k_2-1, \ldots, k_r) + \cdots + R(k_1, k_2, \ldots, k_r-1)$$

Let $n = R(k_1-1, k_2, \ldots, k_r) + R(k_1, k_2-1, \ldots, k_r) + \cdots + R(k_1, k_2, \ldots, k_r-1)$, for brevity. Consider a vertex v in a coloring of a K_n. v has $n-1$ incident edges.

Suppose that v has fewer than $R(k_1-1, k_2, \ldots, k_r)$ incident edges in color 1, fewer than $R(k_1, k_2-1, \ldots, k_r)$ incident edges in color 2, and so forth up to $R(k_1, k_2, \ldots, k_r-1)$ incident edges in color r. Then in total v could have at most

$$(R(k_1-1, k_2, \ldots, k_r)-1) + (R(k_1, k_2-1, \ldots, k_r)-1) + \cdots + (R(k_1, k_2, \ldots, k_r-1)-1) = n-r$$

incident edges in total. Since $r > 1$ and n is incident on exactly $n-1$ edges, this is impossible. Thus, there is *some* i such that v is incident on $R(k_1, k_2, \ldots, k_i-1, \ldots, k_r)$ edges in color i.

Now consider the induced subgraph H among the non-v endpoints of these edges in color i. Since H is a complete graph on $R(k_1, k_2, \ldots, k_i-1, \ldots, k_r)$ vertices, it must contain either a K_{k_1} in color 1, a K_{k_2} in color 2, or so forth, with the exception that it is only guaranteed to contain a K_{k_i-1} in color i. However, if H contains a K_{k_i-1} in color i, we can take that K_{k_i-1} in color i together with v, to which it is adjacent via edges in color i, to get a K_{k_i} in color i. Thus, we know that either G contains a K_{k_j} in color j for *some* j, so $n = |G| \geq R(k_1, k_2, \ldots, k_r)$.

This bound can be fairly trivially improved by applying the same argument to $n = R(k_1, k_2, \ldots, k_r) \leq R(k_1-1, k_2, \ldots, k_r) + R(k_1, k_2-1, \ldots, k_r) + \cdots + R(k_1, k_2, \ldots, k_r - 1) - r + 2$; a less trivial but not very large improvement can be crafted by mimicing the parity argument seen in class.

(b) **(5 points)** *Prove by induction on the number r of colors that*

$$2^r < R(\underbrace{3, 3, 3, \ldots, 3}_{r \text{ times}}) \leq 3r!$$

For brevity, let us refer to $R(\underbrace{3, 3, 3, \ldots, 3}_{r \text{ times}})$ as $R(3^r)$. For the base case, one can note the known fact that $R(3^2) = R(3,3) = 6$ and $4 < 6 \leq 3 \cdot 2!$.

For the inductive step we we will begin by showing that $2^r < R(3^r)$. By our inductive hypothesis, $2^{r-1} < R(3^{r-1})$, so there is an $(r-1)$-coloring of a $K_{2^{r-1}}$ without a monochromatic K_3. We can construct an r-coloring of a K_{2^r} without a monochromatic K_3 as such: take two copies A and B of $K_{2^{r-1}}$ and color each of them according to the aforementioned $(r-1)$-coloring; now use the r-th color to color every edge between the two copies. This is a coloring of K_{2^r} which does not contain a monochromatic K_3: in neither A nor B is there a K_3 in any of the first $r-1$ colors by construction, and there can't be a monochromatic K_3 in the first $r-1$ colors lying between them because every edge between them is in color r. Lastly, there is no K_3 in color r because the edges in color r form a complete bipartite graph by construction. Thus, there is a coloring of K_{2^r} with r colors which contains no monochromatic K_3, so $2^r < R(3^r)$.

To show the upper bound, that $R(3^r) \leq 3n!$, we need to prove that $R(3^r) \leq rR(3^{r-1})$; the inductive hypothesis will then tell us that $rR(3^{r-1}) \leq r(3(r-1)!) = 3r!$. In order to prove this, let us consider a vertex v in a $K_{rR(3^{r-1})}$. Since it has $rR(3^{r-1}) - 1$ incident edges in r colors, the pigeonhole principle will tell us that it has at least $R(3^{r-1})$ incident edges in a single color. Let us call that color "red" for simplicity. Every edge among the $R(3^{r-1})$

endpoints of these edges must be colored in one of the r different colors. If we color *any* edges red, then our coloring will contain a red K_3, using this edge and two edges from its endpoitns to v. If, on the other hand, we color no edge red, then we are coloring the edges among $R(3^{r-1})$ points using $r-1$ colors, which by the definiton of $R(3^{r-1})$ is guaranteed to contain a monochromatic K_3. Thus, any coloring with r colors of the edges among $rR(3^{r-1})$ vertices is guaranteed to contain a monochromatic K_3, so $R(3^r) \le rR(3^{r-1})$.

(c) **(5 points)** *Prove that for* $0 < p < 1$, *if* $\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} < 1$, *then* $R(k,\ell) > n$.

Let the edges of K_n be colored independently at random, but without equal chances of being red or blue, i.e., for each edge, color it red with probability p, and blue with probability $1-p$. For a subset S of $V(K_n)$ with $|S| = k$, let A_S represent the event that the elements of S are the vertices of a red K_k, and for a subset T with $|T| = \ell$, let B_T represent the event that the elements of S are the vertices of a blue K_ℓ. For a specific S, since each of the $\binom{k}{2}$ edges among vertices of S has exactly a probability p of being red, then the probability that the edges among the elements of S are all red (forming a red clique) is $p^{\binom{k}{2}}$. Likewise, the probability that all the edges among a specific set of ℓ vertices are blue is $(1-p)^{\binom{\ell}{2}}$. These are respectively the probabilities of the events A_S and B_T.

Now, the event that there is a red K_k or blue K_ℓ somewhere in the K_n occurs only if some A_S or B_T occurs: in other words, this event occurs only if at least one A_S or B_T occurs. The probability that at least one of a family of events occurs is no more than the sum of their individual probabilities (by inclusion-exclusion, essentially), and thus:

$$\begin{aligned}
Pr(\text{red } k\text{-clique or blue } \ell\text{-clique}) &\le \sum_{|S|=k} Pr(A_S) + \sum_{|T|=\ell} Pr(B_T) \\
&\le \sum_{|S|=k} p^{\binom{k}{2}} + \sum_{|T|=\ell} (1-p)^{\binom{\ell}{2}} \\
&\le \binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} < 1
\end{aligned}$$

so the probability that a random coloring contains a red k-clique or blue ℓ-clique is less than one, so the probability that a random coloring does not contain one of these structures is greater than zero: that is to say, an actual possibility! So, some coloring of K_n contains no red K_k or blue K_ℓ, so $n < R(k,\ell)$.

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4. **(5 point bonus)** *Prove that if* H *is 2-connected, then for any value of* n *every extremal* H*-free graph* G *with* n *vertices (that is,* $|G| = n$, $\|G\| = \text{ex}(n,H)$, *and* $H \not\subseteq G$*) is connected.*

Let H be 2-connected, and suppose G is a non-connected H-free graph. Let G have components C_1 and C_2, and let $e = \{u,v\}$ for $u \in C_1$, $v \in C_2$. We shall see that $G+e$ is H-free: definitionally G is H-free, so any copy of H appearing in $G+e$ must use e; thus, any subgraph of G isomorphic to H would have vertices in both C_1 and C_2. We know any 2 vertices of a 2-connected graph lie on a cycle, so such an H implies that $G+e$ contains a cycle with vertices in both C_1 and C_2; thus G would contain a path from C_1 to C_2, violating disconnectedness. Thus, by contradiction, $G+e$ is also H-free, which means that G could not have been a *maximal* H-free graph; thus $|G| < \text{ex}(n,H)$.