This problem set is due at the beginning of class on *April 6*. Below, "graph" means "simple finite graph" except where otherwise noted.

1. **(10 points)** Complete the proof that the Harary graphs are $k$-connected. You may use the case presented in class of even values of $k$, either by citation or imitation.

   (a) **(5 points)** Show that $H_{n,k}$ is $k$-connected for even $n$ and odd $k$. This graph is symmetric, so you may specifically demonstrate connectedness between $v_1$ and an arbitrary $v_i$ in $H_{n,k} - S$ for $v_1, v_i \notin S$ and $|S| < k$.

   (b) **(5 points)** Show that $H_{n,k}$ is $k$-connected for odd $n$ and odd $k$. This graph is not symmetric, so you must distinguish between the cases where $v_1 \in S$ and $v_1 \notin S$.

2. **(15 points)** Answer the following questions about Turán numbers and extremal graphs.

   (a) **(5 points)** Show that the extremal number $\mathrm{ex}(n, K_{1,r})$ is $\frac{(r-1)n}{2}$ if $r$ is odd or $n$ is even, and $\frac{(r-1)n-1}{2}$ if $r$ is even and $n$ is even.

   (b) **(5 points)** Find a formula for $\mathrm{ex}(n, P_4)$ by finding necessary conditions for a graph to not contain a $P_4$ (i.e. a path with 4 vertices and 3 edges). Exploration on small values of $n$ may be helpful.

   (c) **(5 points)** Prove that for nontrivial $H$, the asymptotic density of any Turán graph has the bounds: $\frac{\chi(H)-2}{\chi(H)-1} \le \lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} \le \frac{|H|-2}{|H|-1}$

3. **(10 points)** Answer the following questions about Ramsey numbers.

   (a) **(5 points)** Show that the $r$-color Ramsey number $R(k_1, k_2, \dots, k_r)$ is subject to the recurrence inequality:

   $$R(k_1, k_2, \dots, k_r) \le R(k_1 - 1, k_2, \dots, k_r) + R(k_1, k_2 - 1, \dots, k_r) + \cdots + R(k_1, k_2, \dots, k_r - 1)$$

   (b) **(5 points)** Prove by induction on the number $r$ of colors that

   $$2^r < R(\underbrace{3, 3, 3, \dots, 3}_{r \text{ times}}) \le 3r!$$

   (c) **(5 points)** Prove that for $0 < p < 1$, if $\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} < 1$, then $R(k, \ell) > n$.

4. **(5 point bonus)** Prove that if $H$ is 2-connected, then for any value of $n$ every extremal $H$-free graph $G$ with $n$ vertices (that is, $|G| = n$, $\|G\| = \mathrm{ex}(n, H)$, and $H \not\subseteq G$) is connected.

> Imagine an alien force, vastly more powerful than us, demanding the value of $R(5,5)$ or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6,6)$. Then we should attempt to destroy the aliens. —Paul Erdős