1. **(6 of 12 students attempted this)** *Answer the following questions related to bound-subverting examples:*

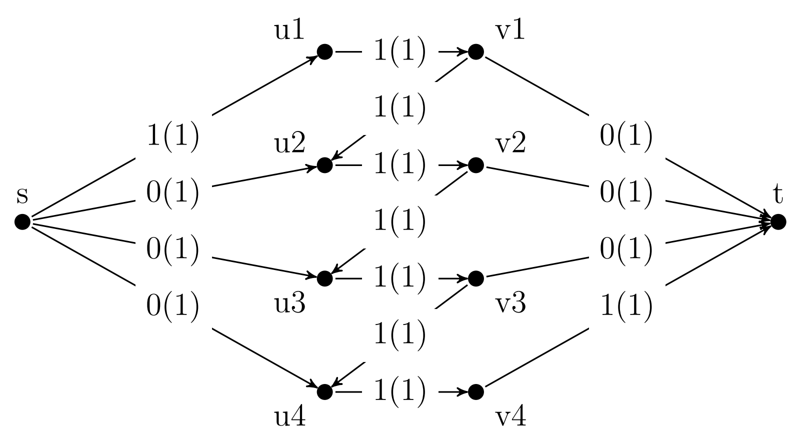
   (a) *It is known that the connectivity $\kappa(G)$ of a graph G is bounded above by the minimum degree $\delta(G)$. Describe a method of constructing a connected graph with arbitrarily large δ but with a fixed, small κ.*

   One easy example is 2 K_ns connected by a single edge. This is connected, and has minimum degree n, but it has connectivity 1, since removing either of the endpoints of the bridge will disconnect the graph.

   (b) *Let a flow f on a weighted digraph D be called* addition-maximal *if there is no valid flow $g \neq f$ such that $g(e) \geq f(e)$ for all edges; in other words, there is no valid flow resulting solely from adding flow to f. Describe (with an example, if such is useful) the construction of a weighted digraph and addition-maximal flow f such that $|f| = 1$ but D has maximal flow k for an arbitrarily large integer k.*

   Consider the following construction: we have source s, sink t, and intermediate vertices $u_1, \ldots, u_k$ and $v_1, \ldots v_k$. Our edges will be (s, u_i), (u_i, v_i), (v_i, u_{i+1}), and (v_i, t) for all i, and we will have capacity 1 on each edge. Suppose f has flow of 1 on the edges (s, u_1), (u_i, v_i), (v_i, u_{i+1}), (v_k, t), and flow 0 on every other edge. Then $|f| = 1$, but the maximum flow on this graph is clearly k, since we could route flow 1 through each of the k paths $s \to u_i \to v_i \to t$. Furthermore, f cannot be improved simply by adding flow, since all of the forward paths (u_i, v_i) are fully utilized.

   Below is an exhibit of how this graph would look in the specific case of $k = 4$:



2. **(4 of 12 students attempted this)** *Let G be a graph containing a cycle C, such that there is a path P of length k between two vertices of G. Show that G contains a cycle of length $\sqrt{k}$. (Hint: give a name to the number of times the P intersects C, and find cycles whose size is dictated by this parameter)*

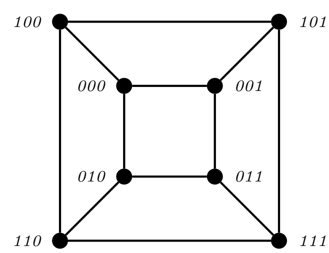
   Let us consider the aforementioned path of length k: P clearly intersects C at least twice, at its endpoints, but let us say that it has $r + 1$ points of intersection with C. We shall see that G must thus contain a cycle of length at least r and a cycle of length at least $\frac{k}{r}$.

The first of the above assertions is easy: since P intersects C in $r+1$ points, C contains at least $r+1$ points, so the length of the cycle C is atleast r (actually, at least $r+1$).

The second assertion is subtler: let us denote the points of P's intersection with C, sequentially along the length of P, as v_0, v_1, v_2,...,v_r. Now let us denote the fragment of P connecting v_{i-1} to v_i as P_i. So P, a path of length k, is constructed by splicing together the r paths $P_1, P_2, \ldots, P_r$; thus, at least one of these P_i has length $\frac{k}{r}$. Now, since P_i only intersects C in its endpoints v_{i-1} and v_i, one can splice P_i with a section of C to form a cycle of length at least $\frac{k}{r}$.

Given that G has cycles of length at least r and $\frac{k}{r}$, we know that since $r \cdot \frac{k}{r} = k$, at elast one of r and $\frac{k}{r}$ must be at least $\sqrt{k}$.

3. **(9 of 12 students attempted this)** *Let the* cube graph Q_k *be the graph whose vertices are bitstrings of length k, such that v is adjacent to u if and only if their bitstrings differ in exactly one bit. Q_3 is shown below.*



Note that Q_k can also be constructed recursively, by taking two copies of Q_{k-1} and connecting their associated vertices.

(a) **(3 points)** *Prove that Q_k is bipartite for all k.*

Let a bitstring be called odd if it contains an odd number of 1s, and even if it contains an odd number of 1s. If $u \sim v$, then the bitstrings u and v differ in exactly one bit, so u has exactly one more or less 1 than v, so if v is even, u must be odd, and vice versa. Thus, all adjacencies are between vertices of opposite parity. If we partition the vertices by parity, we have two parts with no adjacencies within a part, so Q_k is bipartite.

(b) **(3 points)** *Determine, with justification, the values of k for which Q_k is eulerian.*

Since any bitstring is adjacent to the bit formed by "toggling" any particular position, it is easy to see that the degree of every vertex in Q_k is k, since there are k potential bits to flip and find adjacent bitstrings. Q_k can easily be seen to be connected, so it is eularian if and only if every vertex has even degree, or in other words, when k is even.

(c) **(4 points)** *Show that Q_k is k-connected.*

This is easiest to do by induction, and with Menger's Theorem: we shall show that between any two bitstrings of length k there are k independent paths, inducting on k. The case $k=1$ is quite trivial; there is one path in Q_1 from 0 to 1.

Now, we shall inductively demonstrate the existence of k independent paths between bitstrings u and v. Let u' and v' represent the bitstrings of length $k-1$ formed by dropping the last bit of u and v respectively. There are two (fairly similar) cases we shall address.

If u and v have the same last bit: by our inductive assumption there are $k-1$ independent paths in Q_{k-1} from u' to v'; tack the last bit on these to get $k-1$ independent paths in Q_k. Finally, produce a kth path by first flipping the final bit, following any of the $k-1$ previously mentioned independent paths with the final bit reversed, and then reverse the final bit back.

If u and v have different last bit: by our inductive assumption there are $k-1$ independent paths in Q_{k-1} from u' to v'; for all but one of these, let us produce independent paths in Q_k by using the last bit from u for the first and second position, then include a step flipping this bit, and continue the path from Q_{k-1} with the final bit from v. To get our last two independent paths, let us take the one unused path from Q_{k-1} and construct two paths as follows: in one, follow the path in Q_{k-1} from u' to v' with the last bit from u tacked on, and as our last step, flip the last bit to get to v; for our other path, let us flip the last bit immediately to reach a point distinct from u, and then follow our Q_{k-1} path with this flipped final bit to get to v.

4. **(11 of 12 students attempted this)** *Prove the following facts about matchings.*

   (a) **(5 points)** *If $|G|$ is even and $\kappa(G) \geq \frac{|G|}{2}$, then G has a perfect matching.*

   We shall prove this by showing this graph satisfies the conditions of Tutte's Theorem. When $|S| = 0$, since $\kappa(G) > 0$, $G - S$ is connected and even, and thus has zero odd components. When $1 \leq |S| < \frac{|G|}{2}$, by $|S|$-connectivity $G - S$ will be connected, and thus have no more than 1 odd component. And when $|S| \geq \frac{|G|}{2}$, $G - S$ has no more than $\frac{|G|}{2}$ vertices, and thus, even if every single one of those vertices was isolated, would have no more than $\frac{|G|}{2}$ odd components. Thus, in every case, $G - S$ has no more than $|S|$ odd components, so by Tutte's Theorem, G has a perfect matching.

   (b) **(5 points)** *If $G = (A, B)$ is bipartite and k-regular for some $k \geq 1$, then G has a perfect matching.*

   We shall show that this graph satisfies the condition's of Hall's Theorem. Consider some $S \subset A$ or $S \subset B$. S is incident on some set T of edges, and by k-regularity, $|T| = k|S|$. Clearly each of these edges has an endpoint in $N(S)$, so T is a subset of the set T' of edges incident on $N(S)$; again by k-regularity, $|T'| = k|N(S)|$. Since $T \subseteq T'$, $|T| < |T'|$, so $k|S| \leq k|N(S)|$, from which the Hall Criterion follows, so there exist A-perfect and B-perfect matchings; thus there is a perfect matching.

5. **(6 of 12 students attempted this)** *Prove that if a graph G with $|G| > 4$ is 3-regular, then the connectivity $\kappa(G)$ and edge-connectivity $\kappa'(G)$ are equal.*

An easy way to prove this is by Menger's Theorem: it will follow immediately if we show that paths are edge-disjoint if and only if they are independent, since if the sets of edge-disjoint and independent paths are equinumerous, so are minimal vertex-separating and edge-separating sets by the two different versions of Menger's Theorem.

It is easy to see that paths in a 3-regular graph will be independent if and only if they are edge-disjoint. In one direction this implication is easy; independent paths must be edge-disjoint, since not sharing vertices makes it impossible to share edges. On the other hand, suppose P_1 and P_2 were edge-disjoint but shared an internal vertex w; since P_1 must arrive at and depart w, it uses two edges incident on w; likewise P_2 uses two edges incident on w, and since P_1 and P_2 are edge-disjoint, all these edges must be distinct, requiring that w have degree of at least 4, contradicting 3-regularity.

Thus, since collections of edge-disjoint paths and collections of vertex-disjoint paths are identical (and in particular the maximal specimens of each are of the same size), by Menger's theorem the minimal edge- and vertex-separating sets for any two vertices are the same size.

This theorem could also be proven without recourse to Menger's Theorem by explicitly constructing an edge-separating set of the same size as any vertex-separating set: if $G - S$ is disconnected into parts A and B, then every element of S either has three neighbors in one part (in which case it is not necessary for separation, and S is not actually a minimum separating set) or 2 neighbors in one and one neighbor in the other, in which case removing the edge which has unique function will serve the same separatory purpose as removal of the vertex does. The proof of this is a bit ungainly but not difficult.

6. **(10 of 12 students attempted this)** *Let the* cone $C(G)$ *of* G *be a graph such that* $V(C(G)) = V(G) \cup \{c\}$ *and* $E(C(G)) = E(G) \cup \{\{v, c\} : v \in V(G)\}$*; that is, we add one vertex and edges from that vertex to every other vertex of the graph. Prove the following facts about the cone.*

   (a) **(4 points)** $\kappa(C(G)) = \kappa(G) + 1$.

   Using Menger's Theorem, if there are k independent paths between vertices u and v in G, there are $k+1$ such paths in $C(G)$, since the path $u \sim c \sim v$ is guaranteed to be independent of any path in G. There cannot be *more* than $k+1$ such paths, since c can only appear in one path, after which the remaining paths must lie in G, and there are only k such paths.

   Alternatively: if G is disconnected by removing a k vertices, it is clear that exactly $k+1$ vertices must be removed to disconnect $C(G)$, since if we removed a set of $k-1$ vertices from G as well as c itself, then since $C(G) - c = G$, this would be identical to removing $k-1$ vertices from G alone, which would not disconnect it, while removing k vertices from G and leaving c untouched would also leave $C(G)$ connected, since c is adjacent to every vertex and thus connects them all. On the other hand, $k+1$ vertices are clearly sufficient, since removing c itself leaves us with G, which k more removals suffice to disconnect.

(b) **(3 points)** *$C(G)$ has a perfect matching if and only if there is a vertex $v \in G$ such that $G - v$ has a perfect matching.*

If $C(G)$ has a perfect matching, then if c is matched with v, removal of both c and v clearly leaves us with a graph with a perfect matching, simply by removing the edge $\{c, v\}$ from the perfect matching already found. Note that $(C(G) - c) - v = G - v$.

Conversely, if some $G - v$ has a perfect matching, the perfect matching in $G - v$ is a matching in $C(G)$, since $G \subset C(G)$, but leaves c and v unmatched. Since $v \sim c$ in $C(G)$, this matching can be augmented to a perfect matching in $C(G)$ simply by adding $\{c, v\}$.

(c) **(3 points)** *For ω representing the clique number, $\omega(C(G)) = \omega(G) + 1$.*

If G contains a clique $v_1, v_2, \ldots, v_k$, then $v_1, v_2, \ldots, v_k, c$ will be a clique in $C(G)$, so it is easy to show that $\omega(C(G)) \geq \omega(G) + 1$. To show the other direction, note that if $C(G)$ contains a clique of size k, the clique either does not contain c, in which case it lies in G and G contains a clique of size k, or it contains c, in which case removal of c from the clique yields a clique of size $k - 1$ lying in G. Thsu $\omega(G) \geq \omega(C(G)) - 1$; these two inequalities together yield the equality sought.