Answer exactly four of the following six questions. *Indicate which four you would like graded!*

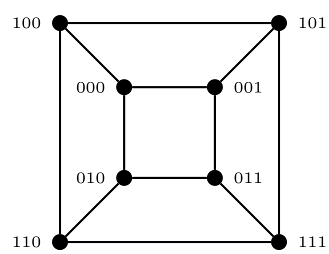
1. **(10 points)** Answer the following questions related to bound-subverting examples:

   (a) **(4 points)** It is known that the connectivity $\kappa(G)$ of a graph G is bounded above by the minimum degree $\delta(G)$. Describe a method of constructing a connected graph with arbitrarily large δ but with a fixed, small κ.

   (b) **(6 points)** Let a flow f on a weighted digraph D be called *addition-maximal* if there is no valid flow $g \neq f$ such that $g(e) \geq f(e)$ for all edges; in other words, there is no valid flow resulting solely from adding flow to f. Describe (with an example, if such is useful) the construction of a weighted digraph and addition-maximal flow f such that $|f| = 1$ but D has maximal flow k for an arbitrarily large integer k.

2. **(10 points)** Let G be a graph containing a cycle C, such that there is a path P of length k between two vertices of G. Show that G contains a cycle of length $\sqrt{k}$. (Hint: give a name to the number of times the P intersects C, and find cycles whose size is dictated by this parameter)

3. **(10 points)** Let the *cube graph* Q_k be the graph whose vertices are bitstrings of length k, such that v is adjacent to u if and only if their bitstrings differ in exactly one bit. Q_3 is shown below.



   Note that Q_k can also be constructed recursively, by taking two copies of Q_{k-1} and connecting their associated vertices.

   (a) **(3 points)** Prove that Q_k is bipartite for all k.

   (b) **(3 points)** Determine, with justification, the values of k for which Q_k is eulerian.

   (c) **(4 points)** Show that Q_k is k-connected.

4. **(10 points)** Prove the following facts about matchings.

   (a) **(5 points)** If $|G|$ is even and $\kappa(G) \geq \frac{|G|}{2}$, then G has a perfect matching.

   (b) **(5 points)** If $G = (A, B)$ is bipartite and k-regular for some $k \geq 1$, then G has a perfect matching.

5. **(10 points)** Prove that if a graph G with $|G| > 4$ is 3-regular, then the connectivity $\kappa(G)$ and edge-connectivity $\kappa'(G)$ are equal.

6. **(10 points)** Let the *cone* $C(G)$ of G be a graph such that $V(C(G)) = V(G) \cup \{c\}$ and $E(C(G)) = E(G) \cup \{\{v, c\} : v \in V(G)\}$; that is, we add one vertex and edges from that vertex to every other vertex of the graph. Prove the following facts about the cone.

   (a) **(4 points)** $\kappa(C(G)) = \kappa(G) + 1$.

   (b) **(3 points)** $C(G)$ has a perfect matching if and only if there is a vertex $v \in G$ such that $G - v$ has a perfect matching.

   (c) **(3 points)** For ω representing the clique number, $\omega(C(G)) = \omega(G) + 1$.