1. **(8 students attempted this problem)** *Suppose that G is a simple graph that contains two edges whose removal destroys all cycles in G. Prove that G is planar.*

   The easiest approach is to prove the contrapositive via Kuratowski's Theorem: if G is nonplanar, then G still contains cycles after removal of any two edges. We know G is nonplanar only if G contains a subdivision of $K_{3,3}$ or K_5. A subdivision of $K_{3,3}$ with n vertices will contain $n+3$ edges; a subdivision of K_5 with n vertices will contain $n+5$ edges. In order for a graph to be cycle-free, it must have fewer edges than vertices (a tree being such a graph with as many edges as possible), so one would have to remove at least 4 edges from a nonplanar graph to bring the edge count on its $K_{3,3}$ or K_5 subdivision down low enough.

   Another approach may be by explicit tree-manipulation. If $G - e_1 - e_2$ is cycle-free, this means that $F = G - e_1 - e_2$ is a forest. Let us consider the reinsetion of these edges into the forest. One edge can be added easily to any planar representation of a forest, since there are no cycles to serve as face boundaries and separate one vertex from another. If this edge does not create a cycle, then the second edge can be trivially added by the same rule. If, on the other hand, the first edge adds a cycle, then we have a planar representation with exactly one cycle (since a path between the two endpoints of a putative added edge in a tree is unique, adding this edge produces a *unique* cycle). Thus, in this case $F + e_1$ consists of one or more trees as well as a component with a single cycle. We may orient all other features of this graph to point out of the cycle; now the edges subdivide the plane into two parts — the interior of the cycle and the exterior. By our edge-orientation, every vertex lies on the exterior or boundary of the cycle, so the second edge can be added in accordance with planarity, by tracing it through the cycle's exterior.

   Note: this result is also true for removal of three edges; the Kuratowski's Theorem approach is sufficient to show that with no modification whatsoever. That result can also be shown by direct arrangement of a tree, but it's quite fiddly.

2. **(11 students attempted this problem)** *Let $v_1 \sim v_2 \sim v_3 \sim \cdots \sim v_n$ be the longest path in a simple graph G. Show that $\chi(G) \leq n$.*

   One easy approach to this is by observing that a greedy coloring induces a path through every color used. Suppose a greedy coloring c uses k colors. Then there is some u_1 such that $c(u_1) = k$; in order for using color k at u_1 to be necessary, u_1 must be adjacent to vertices in every color from 1 to $k-1$; specifically, there must be a u_2 such that $u_1 \sim u_2$ with $c(u_2) = k-1$; by the same token, there must be a u_3 such that $u_2 \sim u_3$ with $c(u_3) = k-2$, and so forth up to $u_{k-1} \sim u_k$ with $c(u_k) = 1$. We thus have a walk $u_1 \sim u_2 \sim \cdots \sim u_k$, and since the vertices are all different colors, they are all distinct, so this walk is a path. But the longest possible path visits n vertices, so we know $k \leq n$. Since we colored G with k colors, we know $\chi(G) \leq k \leq n$.

3. **(6 students attempted this problem)** *Prove that there is a tournament on n vertices with a directed Eulerian tour if and only if n is odd.*

   An explicit construction will suffice to show that such a tournament exists when n is odd: since K_n has degree $n-1$ everywhere, it has an Eulerian tour; such a tour can be oriented simply by tracing it in a specific direction, and the result of such an orientation is a tournament.

When n is even, each vertex is incident on an odd number of edges, and thus has no Eulerian tour even when divested of orientation; thus no orientation can produce a directed Eulerian tour.

4. **(4 students attempted this problem)** *The* box product $G \boxdot H$ *of two graphs G and H is a graph with vertices represented by ordered pairs (u_G, u_H) where $u_G \in G$ and u_H in H. (u_G, u_H) is adjacent to (v_G, v_H) if either $u_G = v_G$ and $u_H \sim v_H$ or $u_G \sim v_G$ and $u_H = v_H$.*

   (a) **(5 points)** *Prove that $\chi(G \boxdot H) \leq \chi(G)\chi(H)$.*

   If we color G with the colors $\{1, \ldots, r\}$ and H with $\{1, \ldots, s\}$, we can color $G \boxdot H$ with the rs colors represented by the ordered pairs (i, j) with $1 \leq i \leq r$ and $1 \leq j \leq s$. An extremely naïve coloring will suffice: let (u_G, u_H) be colored with an ordered pair consisting of the colors of u_G and u_H respectively. Then, for $(u_G, u_H) \sim (v_G, v_H)$, either $u_G \sim v_G$ or $u_H \sim v_H$, so in one or the other coordinate these points are guaranteed to have different colors, ensuring that our coloring is legal.

   Note: one can actually do much better than this; in fact $\chi(G \boxdot H) = \max(\chi(G), \chi(H))$; let us assume $k = \max(\chi(G), \chi(H))$ for simplicity. Now let us construct k-colorings c_G and c_H on each of G and H, and color the vertex (v_G, v_H) with a color congruent to $c_G(v_G) + c_H(v_H)$ modulo k. It is not hard to show that this coloring assigns distinct colors to adjacent vertices.

   (b) **(5 points)** *Prove that $\chi'(G) + \chi'(H) - 2 \leq \chi'(G \boxdot H) \leq \chi'(G) + \chi'(H) + 1$.*

   Since edge-coloring numbers are almost exclusively determined by maximum degree (as guaranteed by Vizing's Theorem), we shall explore the maximum degree. Since (u_G, u_H) has as neighbors all (u_G, v_H) with $u_H \sim v_H$ and (v_G, u_H) with $u_G \sim v_G$, it is clear that $d_{G \boxdot H}((u_G, u_H)) = d_G(u_G) + d_H(u_H)$, and specifically that $\Delta(G \boxdot H) = \Delta(G) + \Delta(H)$.

   Invoking Vizing's Theorem several times, we know $\chi'(G) - 1 \leq \Delta(G) \leq \chi'(G)$ and $\chi'(H) - 1 \leq \Delta(H) \leq \chi'(H)$, so

   $$\chi'(G) + \chi'(H) - 2 \leq \Delta(G \boxdot H) \leq \chi'(G) + \chi'(H)$$

   so the lower bound on $\chi'(G \boxdot H)$ is, by Vizing's Theorem, $\Delta(G \boxdot H) \geq \chi'(G) + \chi'(H) - 2$; the upper bound is $\Delta(G \boxdot H) + 1 \leq \chi'(G) + \chi'(H) + 1$.

   Note: this upper bound could be reduced slightly by an explicit construction. In fact, $\chi'(G \boxdot H) \leq \chi'(G) + \chi'(H)$, since we could use $\chi'(G)$ colors to color all the edges corresponding to edges of G, and $\chi'(H)$ other colors to color all the edges corresponding to edges of H.

5. **(12 students attempted this problem)** *The cube Q_4 consists of sixteen vertices associated with the sixteen bitstrings* $0000, 0001, \ldots, 1111$. *Two vertices are adjacent if they differ in exactly one bit. Prove that Q_4 is nonplanar.*

Note that each vertex has degree 4, and that this graph is bipartite. We have 32 edges in total, so if this graph were planar it would need to have $\|Q_4\| + 2 - |Q_4| = 18$ faces by Euler's formula. Since this graph is bipartite, each face would need to have length 4 or more, giving $\|Q_4\| \geq \frac{18 \cdot 4}{2} = 36$, which is not true.

There are also several explicit $K_{3,3}$ subdivisions in Q_4, one of which is given here: take 0000, 0011, and 0110 as one part, 0001, 0010, and 0100 as the other, and connect them via 9 independent paths; 7 of the paths are simple adjacencies, and the last two are $0011 \sim 1011 \sim 1111 \sim 1110 \sim 1100 \sim 0100$ and $0110 \sim 0111 \sim 0101 \sim 0001$.

Note: Q_n is easily shown to be planar by explicit drawing for all $n \leq 3$; since Q_4 is a subgraph of every larger cube, Q_n is nonplanar for all $n \geq 4$.

6. **(7 students attempted this problem)** *Prove by construction that for $n > 2k$, $\mathrm{ex}(C_{2k}, n) \geq (k-1)(n-k+1)$.*

The graph $K_{k-1,n-k+1}$ has $(k-1)(n-k+1)$ edges; all we need to do is show that it is C_{2k}-free. Any cycle in a bipartite graph alternates between parts; thus half the vertices in a cycle must come from each part. Specifically, if a C_{2k} lies in a bipartite graph, k vertices of the cycle lie in each part. Since one part of $K_{k-1,n-k+1}$ has fewer than k vertices, it cannot contain a C_{2k}.

Note: exact values of $\mathrm{ex}(C_{2k}, n)$ are not known in general. There are advanced general results (Erdős and Stone, '46, which appears in Diestel) showing that the asymptotic density of a family of H-free graphs with bipartite H is zero; this merely tells us that $\mathrm{ex}(C_{2k}, n)$ is, over the long term, smaller than a quadratic. Erdős and Simonovits conjectured that it tends towards $\frac{1}{2}n^{1+\frac{1}{k}}$. A 2005 result of Füredi, Naor, and Verstraëte gives $\mathrm{ex}(C_{2k}, n) > \frac{3\sqrt{5}-6}{(\sqrt{5}-1)^{4/3}}n^{4/3}$, and this is the most explicit progress to date on this problem.