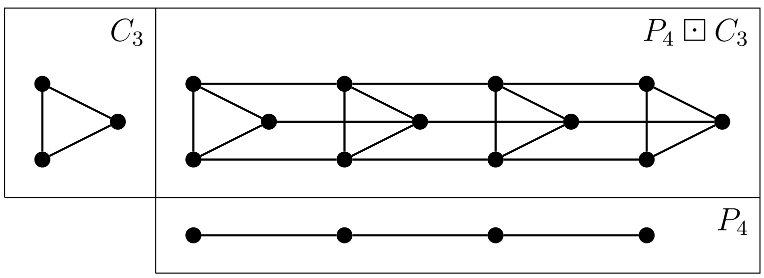
Answer exactly four of the following six questions. *Indicate which four you would like graded!*

1. **(10 points)** Suppose that G is a simple graph that contains two edges whose removal destroys all cycles in G. Prove that G is planar.

2. **(10 points)** Let $v_1 \sim v_2 \sim v_3 \sim \cdots \sim v_n$ be the longest path in a simple graph G. Show that $\chi(G) \leq n$.

3. **(10 points)** Prove that there is a tournament on n vertices with a directed Eulerian tour if and only if n is odd.

4. **(10 points)** The *box product* $G \boxdot H$ of two graphs G and H is a graph with vertices represented by ordered pairs (u_G, u_H) where $u_G \in G$ and u_H in H. (u_G, u_H) is adjacent to (v_G, v_H) if either $u_G = v_G$ and $u_H \sim v_H$ or $u_G \sim v_G$ and $u_H = v_H$. As an example, we see the result of taking a box product of C_3 and P_4 below:



   (a) **(5 points)** Prove that $\chi(G \boxdot H) \leq \chi(G)\chi(H)$.

   (b) **(5 points)** Prove that $\chi'(G) + \chi'(H) - 2 \leq \chi'(G \boxdot H) \leq \chi'(G) + \chi'(H) + 1$.

5. **(10 points)** The cube Q_4 consists of sixteen vertices associated with the sixteen bit-strings $0000, 0001, \ldots, 1111$. Two vertices are adjacent if they differ in exactly one bit. Prove that Q_4 is nonplanar.

6. **(10 points)** Prove by construction that for $n > 2k$, $\text{ex}(C_{2k}, n) \geq 2k(n - 2k)$.