1 Local and Global 2-connectedness

1.1 Block structure, concluded

Previously we saw that the block structure of a connected graph was connected; today we shall see that it is in fact a tree.

Proposition 1. *Given a connected graph G with block diagram $B(G)$, $B(G)$ is acyclic.*

Proof. Suppose $B(G)$ contains the minimal cycle $B_1 \sim v_1 \sim B_2 \sim v_2 \sim B_3 \sim \cdots \sim B_n \sim v_n \sim B_1$. We shall show that the subgraph $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n$ of G is 2-connected, violating the maximality condition of the blocks B_i.

Note that B is connected, since if $u \in B_i$ and $v \in B_j$, and without loss of generality $i \leq j$, we may construct a walk from u to v via one of the two following scenarios. If $i = j$, then we simply construct a path inside of B_i between the two. If $i < j$, then we build our walk by placing the following paths end-to-end: the path from u to v_i guaranteed by connectivity of B_i, the path from v_i to v_{i+1} guaranteed by connectivity of B_{i+1}, and so forth up to the path from v_{j-2} to v_{j-1} guaranteed by connectivity of B_{j-1}, and then a path from v_{j-1} to v guaranteed by connectivity of B_j.

Now, we shall see that such a walk will remain even if an arbitrary vertex w other than u or v themselves is removed from B. There are several possible vertices w, msot of which have no effect on the construction above: if $w \in B_k$ but $w \neq v_{k-1}, v_k$, then by 2-connectivity of individual blocks, $B_k - w$ is still connected, and all of the named vertices mentioned in the procedure above are still under consideration. Likewise, removal of any v_k with $k < i$ or $k \geq j$ will have no effect on the walk-construction procedure above.

Our only concern, then, is removal of a vertex v_k with $i \leq k < j$. In this case, we can construct a walk with the opposite routing arond the cycle, by placing the following paths end-to-end: the path from u to v_{i-1} guaranteed by connectivity of B_i, the path from v_{i-1} to v_{i-2} guaranteed by connectivity of B_{i-1}, and so forth through the path from v_2 to v_1 guaranteed by connectivity of B_2, and then onto the path from v_1 to v_k guaranteed by connectivity of B_1, continuing to descend to the path from v_{j+1} to v_j guaranteed by connectivity of B_{j+1}, and then a path from v_j to v guaranteed by connectivity of B_j.

Thus, B is 2-connected, so the ostensible blocks within it are not blocks, contradicting the possibility of a cycle of blocks. □

We thus know that a connected graph consists of a number of 2-connected graphs joined along a "skeleton", which is in fact a tree. We can, if we like, use tihs to build a block-structural version of the structure theorem for 1-connected graphs:

Proposition 2. *For any connected graph G, there is a sequence of connected subgraphs $G_1 \subset G_2 \subset \cdots \subset G_n = G$ such that:*

- *G_1 is 2-connected or a single edge.*
- *For $i > 1$, $G_i = G_{i-1} \cup H_i$, where H_i is 2-connected or a single edge, and $|V(G) \cap V(H_i)| = 1$.*

Proof. This follows immediately from the tree structure of the block diagram and the structural theorem on trees. □

1.2 The structure of 2-connected graphs

Now that we have shown arbitrary connected graphs have 2-connected substructure, the question of what that substructure looks like becomes a matter of interest. One observation we can make easily is that every vertex is on a cycle:

Proposition 3. *If G is 2-connected and v is a vertex of G, then there is a cycle in G containing v.*

Proof. We know from a previous result that $\kappa(G) \leq \delta(G)$; since $\kappa(G) \geq 2$ in this case, it follows that $\delta(G) \geq 2$, so, specifically, $d_G(v) \geq 2$. Thus v has at least two neighbors, u_1 and u_2. Since G is 2-connected, $G - v$ is connected, so there is a path $u_1 = v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_k = u_2$ in $G - v$; this is obviously a path in G such that no $v_i = v$. Then, we may easily construct the cycle $v \sim v_0 \sim v_1 \sim \cdots \sim v_k \sim v$ which contains v. □

Note that the converse of this proposition is *not* true. We may join two C_3s at a single vertex and form a graph in which every vertex lies in a cycle, but the graph itself is not 2-connected.

If we want a necessary-and-sufficient cycle-based criterion for 2-connectedness, we can make use of the following structural theorem.

Theorem 1 (Structure theorem for 2-connected graphs)**.** *A graph G is 2-connected if and only if it is constructible via the sequence $G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_k = G$ for a sequence defined as such:*

- *G_0 is a cycle.*
- *$V(G_{i+1}) = V(G_i) \cup \{v_{i,1}, v_{i,2}, \ldots, v_{i,k_i}\}$ for a non-negative integer k, and $E(G_{i+1}) = E(G_i) \cup \{\{u, v_{i,1}\}, \{v_{i,1}, v_{i,2}\}, \{v_{i,2}, v_{i,3}\}, \ldots, \{v_{i,k_i-1}, v_{i,k_i}\}, \{v_{i,k_i}, w\}\}$, where u and w are distinct elements of $V(G_i)$. More concisely: G_{i+1} results by attaching a path to two vertices of G_i.*

Proof. The easy part of this theorem is showing that all graphs so generated are 2-connected. This is easily done by induction: the cycle G_0 is known to be 2-connected, and if we assume that G_i is 2-connected, it follows easily that G_{i+1} is also 2-connected by demonstrating the connectivity of the graphs $G_{i+1} - v$ for various possibilities of v: the two terminal points of the path adjoined to G_i, the new vertices along the path, and the other vertices of G_i.

The difficult (and clever) part is showing that *every* 2-connected graph can be constructed this way. Let G be a 2-connected graph. We know via prior results that there is a cycle somewhere in G. Let this cycle be denoted G_0, and let each G_{i+1} be constructed by adjoining a path which is in $G - G_i$ except for its terminal points to G_i until there are no such paths remaining; denote the end of this sequence by G_k. Our result will follw if we can show that $G = G_k$. Note that by construction G_k is a subgraph of G, but we seek to show it cannot be a proper subgraph.

We start by showing that $V(G) = V(G_k)$. Suppose that they are unequal; then the vertices of G can be partitioned into nonempty $V(G_k)$ and $V(G) \setminus V(G_k)$. By connectivity of G, for arbitrary elements of these two parts of the vertex-set, there is a path between them; this path cannot use vertices solely from a single part, so some edge is between the two parts: we may thus be assured of the existence of $u \in V(G_k)$ and $v \notin V(G_k)$ such that u and v are adjacent in G. Furthermore, since each $|G_i| > 3$, we know G_k has at least two vertices, so let some vertex in G_k other than u be denoted w. By 2-connectivity of G, there is a path in $G_k - u$ from v to w: $v = v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_\ell = w$. Since $v_\ell = w \in V(G_k)$, there is some least index i such that $v_i \in V(G_k)$ (clearly ℓ is such an index, but it may not be the least). Then $u \sim v = v_0 \sim v_1 \sim \cdots \sim v_i$ is a path in G whose terminal vertices are in G_k and none of whose intermediary vertices are: this path may thus be grafted onto G_k to get a larger construction G_{k+1}, contradicting our definition of G_k as an unenlargeable construction. Thus, our assumption that $V(G_k) \neq V(G)$ leads to to a contradiction, and G_k and G must have the same vertex-set.

Now, we shall prove that $E(G_k) = E(G)$. This is actually quite easy: suppose $\{u, v\} \in E(G) \setminus E(G_k)$. Then $u \sim v$ is a path in G whose terminal vertices are in G_k and whose intermediary vertices are nonexistent; this path may be added to G_k to get a larger construction G_{k+1}, contradicting our unenlargeability condition, as above, so $E(G_k) = E(G)$. □

Together with block structure, this gives us an interesting construction for connected graphs: at each stage of the construction we add either an edge from an extant vertex to a new vertex (a new construction of a trivial block), a cycle incident on a single extant vertex (a new construction of a nontrivial block), or a path between vertices of a cycle (a modification of a previously created nontrivial block).

2 Connectivity and multiple paths

Connectivity relates to routing in ways other than finding vertex-cuts. We find that the *number* of paths between two vertices is greatly affected by connectivity. We shall think of both connectivity and distinct-routability, at present, as local properties, relating to specific pairs of vertices, and then generalize to the graph as a whole.

Definition 1. If u and v are nonadjacent vertices of G, a set $S \subset V(G) - \{u, v\}$ is *$\{u, v\}$-separating* if there is no path in $G - S$ between u and v.

Definition 2. A pair of paths P_0 and P_1 between vertices u and v are *independent* if $E(P_0) \cap E(P_1) = \emptyset$ and $V(P_0) \cap V(P_1) = \{u, v\}$; that is, if P_0 and P_1 are paths which differ in all elements except for their terminal vertices. A collection $\mathcal{P}$ of paths is independent if its elements are pairwise independent.

We will start by observing an easy result which we may, at this point, recognize as a first step towards a min-max result.

Proposition 4. *If u and v are nonadjacent vertices of a graph G, then if S is a $\{u, v\}$-separating set and $\mathcal{P}$ is a collection of independent paths from u to v, then $|S| \leq |\mathcal{P}|$.*

Proof. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$. Clearly $V(P_i) \subseteq V(G)$. However, since $G - S$ is disconnected, $P_i \not\subset G - S$ so $V(P) \nsubseteq V(G) \setminus S$. In order for this to be the case, $V(P)$ must have a nontrivial intersection with S; let $s_i \in S \cap P_i$. Note that since u and v are definitionally not in S, s_i is not equal to either u or v. Thus, for any $j \neq i$, $s_i \notin V(P_i) \cap V(P_j)$, so $s_j \neq s_i$. Thus, $\{s_1, s_2, \ldots, s_k\}$ is a k-element subset of S, so $|S| \geq |\mathcal{P}|$. □

In light of the min-max results we saw with chain-partitions and antichains (Dilworth's Theorem), and set-covers and matchings (The König-Egerváry Theorem), it is probably no surprise that a similar result holds for this particular min-max structure. Unfortunately, it is not easy to prove via the others (that I've found).

Theorem 2 (Menger '27)**.** *If u and v are nonadjacent vertices of a graph G, then there is a $\{u, v\}$-separating set S and collection of independent paths $\mathcal{P}$ from u to v, such that $|S| = |\mathcal{P}|$.*

Proof. The theorem is trivially true when u and v are already disconnected (since both S and $\mathcal{P}$ can be empty then), and when there is a single vertex disconnecting u and v (since then clearly $|\mathcal{P}| \geq 1$ and there is an S such that $|S| = 1$, and the previous proposition guarantees equality).

Let us suppose the theorem is, in general, false. Let k be the minimum size of S; let us, from among the space of counterexamples to this theorem, select those with minimum value of k and from that set select a graph G with as few edges as possible. We shall show that such a G cannot, in fact, exist.

First, we shall notice that no vertex x can be adjacent to both u and v. If there were such an x, then it would necessarily be in S, and likewise some path through x would be in a maximal-sized choice of $\mathcal{P}$, since any $\mathcal{P}$ not utilizing x could be improved by including the path $u \sim x \sim v$. Then $G-x$ would have separating set $S-x$, and at least one fewer independent path than $|\mathcal{P}|$, and would be a counterexample in its own right, but with smaller separating set, contradicting minimality of G.

Suppose S does not consist entirely of neighbors of u, or entirely of neighbors of v. Then we define a new graph G' by replacing the component of $G-S$ containing u by a single vertex u', and joining u' to every vertex in S. This will still require removal of k vertices to separate u' and v, but has fewer edges than G. Thus G' is *not* a counterexample to Menger's Theorem, and has independent paths from u' to v through every single element of S. Thus, looking at the unmodified half of these paths, we know there are independent paths from v to all k elements of S. Reversing our procedure and deflating v this time, we get paths from u to every vertex of S. Pairing these paths off gives us k paths from u to v, contradicting our assumption that G w a Menger's Theorem counterexample.

We are left with one possibility: what if all the elements of S are adjacent to u, or all adjacent to v? Now we end up considering a specific path from u to v in G: let $u \sim x_1 \sim x_2 \sim \cdots \sim x_r \sim v$ be a *shortest* path from u to v in G (so that $r = d(u, v) - 1$). The minimal-length of this path easily shows that x_1 is the unique element of the path adjacent to u and x_r the unique element of the path adjacent to v; furthermore, by nonadjacency of u and v and the nonexistence of vertices adjacent to both u and v, we know $r \geq 2$. Let us consider the graph G' created by removing the edge $\{x_1, x_2\}$ from G. By the minimality condition on G, G' cannot be a counterexample to Menger's Theorem, and since G has no more than $k-1$ independent paths, so does G', and thus G' must have a $\{u, v\}$-separating set of size $k-1$, which we shall call S_0. It thus follows that G has at least two distinct $\{u, v\}$-separating sets of size $k-1$: $S_0 \cup \{x_1\}$ and $S_0 \cup \{x_2\}$. However, we have seen that a separating set must consist either solely of neighbors of u or solely of neighbors of v. Since x_1 is a neighbor of u and not a neighbor of v, it follows that $S_0 \cup \{x_1\}$ must lie in the neighborhood of u; however, since x_2 is not a neighbor of v, $S_0 \cup \{x_2\}$ must lie in the neighborhood of v. Thus, the nonempty set S_0 lies in the neighborhoods of both u and v, contradicting our earlier observation that u and v cannot have mutual neighbors. □

As mentioned, there are equivalencies between this result and the other min-max theorems seen in this course. One can prove Menger's theorem by making use of the König-Egerváry theorem by means of a fairly unintuitive construction; details are in Schrijver's *Combinatorial Optimization.* In contrast, the implication in the opposite direction is quite easy.

Theorem 3 (König-Egerváry, '31)**.** *A bipartite graph G contains a matching M and set-cover S such that $|M| = |S|$.*

Proof by way of Menger. For G consisting of parts A and B, let G' be the graph produced by adding a vertex u adjacent to every vertex of A, and a vertex v adjacent to every vertex of B. Let S be a minimal $\{u,v\}$-separating set. Note that if $\{a,b\} \in E(G)$, then since the path $u \sim a \sim b \sim v$ cannot be in $G - S$, either a or b is in S, so S is a vertex-cover of G.

Let $\mathcal{P}$ be a set of independent paths between u and v in G. For each $P_i \in mathcalP$, denote the first three vertices in P_i by $u \sim a_i \sim b_i$. By independence of the paths, for distinct i and j, $a_i \neq a_j$ and $b_i \neq b_j$. Thus, the set $M = \{\{a_1,b_1\},\{a_2,b_2\},\ldots,\{a_{|\mathcal{P}|},b_{|\mathcal{P}|}\}\}$ consists of edges in G with distinct endpoints; that is, a matching in G, and by construction $|M| = |\mathcal{P}|$.

Making use of Menger's Theorem for the second equality, we can conclude thus that $|M| = |\mathcal{P}| = |S|$. □

There are several other common presentations of Menger's Theorem, besides the one we have shown above. Two of these are actually quite easy to express:

Theorem 4 (Menger '27)**.** *There are at least $\kappa(G)$ independent paths between any two vertices of G.*

Proof. For nonadjacent vertices u and v, this is an easy consequence of the prior presentation of the theorem: a minimal $\{u,v\}$ separating set S disconnects the graph, and since it is a vertex-cutset, it must contain at least $\kappa(G)$ elements. The above theorem shows that there is a set of independent paths $\mathcal{P}$ between u and v such that $|\mathcal{P}| = |S| \geq \kappa(G)$. If u and v are adjacent, note that removal of the edge $\{u,v\}$ form G decreases its connectivity by at most 1, so there are at least $\kappa(G) - 1$ independent paths between u and v in $G - \{u,v\}$, which together with the trivial one-edge path is a set of κ independent paths in G. □

Theorem 5 (Menger '27)**.** *For subsets A and B of $V(G)$, there is a set $\mathcal{P}$ of disjoint paths from vertices of A to vertices of B, and a set S of vertices such that $G - S$ contains no paths from A to B, such that $|S| = |\mathcal{P}|$.*

Proof. This follows immediately from constructing a larger graph with a vertex u adjacent to all vertices of A, and v adjacent to all vertices of B, and invoking the previous presentation of Menger's Theorem on it. □

The following two variants on Menger's Theorem we shall *not* prove at this time:

Theorem 6 (Menger '27)**.** *If u and v are nonadjacent vertices of a graph G, then there is a $\{u,v\}$-separating set of edges S and collection of edge-disjoint paths $\mathcal{P}$ from u to v, such that $|S| = |\mathcal{P}|$.*

Theorem 7 (Menger '27)**.** *There are at least $\kappa'(G)$ edge-disjoint paths between any two vertices of G.*