1 Flows

A logical followup (or predecessor) to Menger's theorem, and its associated discussion of multiple simultaneous routes, is the concept of *flow* in a graph. If we visualize a graph as a system of pipelines between a *source* and a *sink*, then a significant question becomes how to route whatever it is we're transferring to make optimal use of our network.

To properly address flow, we'll actually need to make a few changes to the concept of a graph to generalize it better.

Definition 1. A system $D = (V, E)$ where V is a set of vertices and E is a set of *ordered* pairs of elements of V is called a *directed graph* or *digraph*. The edge (u, v) is said to point *from u to v*.

Many of the concepts defined for graphs have analogues in directed graphs. For instance, a *directed walk* is a sequence of edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, frequently denoted $v_0 \to v_1 \to \cdots v_k$. There are likewise concepts of directed trails and paths, and of connectivity by means of directed paths, called *strong connectivity*. Digraphs also possess twovarieties of the concept of vertex degree: the indegree $d^-(v)$ counting incoming edges, and the outdegree $d^+(v)$ counting outgoing edges. We will develop other, novel ideas associated with directed graphs in greater detail after discussing flows.

Another concept necessary when discussing flow is *weight*, which is an assignment of values to graph features, usually edges, but occasionally vertices:

Definition 2. An *edge-weight* for a (di)graph G is a function $f : E(G) \to \mathbb{R}$, and a *vertex-weight* is a function $f : V(G) \to \mathbb{R}$. A (di)graph together with a weight of either type is called a *weighted (di)graph*.

Weights are usually, although not necessarily, restricted to non-negative values. Weight can represent a number of different edge properties in practice: they can be utilization cost, length of connection (in which case the concept of distance is frequently redefined), quality of connection, or possible throughput.

It is the last of these properties that we shall address with regard to flow. The edges of a graph could be thought of as pipelines for material, while the vertices are stations for management and routing. With this model in mind, a question with clear real-world applications is: what is the largest quantity of material we can transfer this way, and how do we do so?

We shall formally pose this question by indicating which transmission plans are "legal" in the sense of actually representing a viable flow. For semantic simplicity, below we shall refer to the edge-weights in a weighted digraph as "capacities", and denote the capacity of an edge e by $c(e)$; for simplicity, we shall also use $c(u, v)$ to denote capacity on the edge (u, v) instead of the more formal but uglier $c((u, v))$.

Definition 3. Let a weighted digraph G with nonnegative weights have distinguished vertices s and t, known as the *source* and *sink* respectively. Then a *flow* on this graph is a function $f : E(G) \to \mathbb{R}$ such that:

- For each $e \in E(G)$, $0 \le f(e) \le c(e)$.
- For each $v \in V(G) \setminus \{s, t\}$, $\sum_{u \to v} f(u, v) = \sum_{w \leftarrow v} f(v, w)$.

These conditions correspond to obvious physical restrictions: the first statement signifies that each pipeline can carry any amount of material between zero and its capacity, and the second indicates that the total flow into to a single station must equal the flow out of that station, with the exception of the source and sink stations.

Proposition 1. *In a digraph G with source s and sink t, for any flow f, $\sum_{v \leftarrow s} f(s,v) - \sum_{u \to s} f(u,s) = \sum_{u \to t} f(u,t) - \sum_{v \leftarrow t} f(t,v)$.*

Proof. We can enumerate the edges of the graph in two different ways: by selecting their head first, or by selecting their tail first. We shall sum the flow through every edge of the graph via this enumeration:

$$\sum_{u \in V(G)} \sum_{v \leftarrow u} f(u,v) = \sum_{e \in E(G)} f(e) = \sum_{v \in V(G)} \sum_{u \to v} f(u,v)$$

Now, since they are rather special cases, we shall consider s and t separately:

$$\sum_{v \leftarrow s} f(s,v) + \sum_{v \leftarrow t} f(t,v) \sum_{u \in V(G) \setminus \{s,t\}} \sum_{v \leftarrow u} f(u,v) = \sum_{u \to s} f(u,s) + \sum_{u \to t} f(u,t) \sum_{v \in V(G) \setminus \{s,t\}} \sum_{u \to v} f(u,v)$$

We know by the conditions set on the flow that for all $w \neq s,t$, $\sum_{u \to w} f(u,w) = \sum_{v \leftarrow w} f(w,v)$, so the third terms on each side of the above equality are equal. Canceling them out, we get

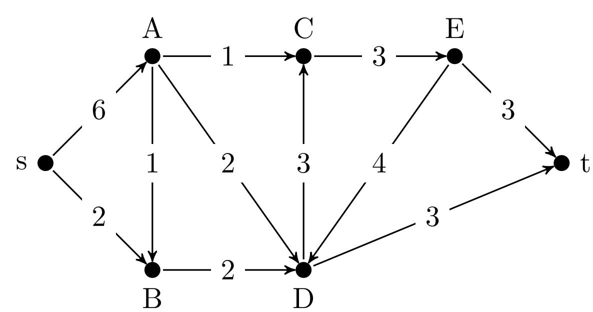
$$\sum_{v \leftarrow s} f(s,v) + \sum_{v \leftarrow t} f(t,v) = \sum_{u \to s} f(u,s) + \sum_{u \to t} f(u,t)$$

which can be rearranged to give the statement of this proposition. □

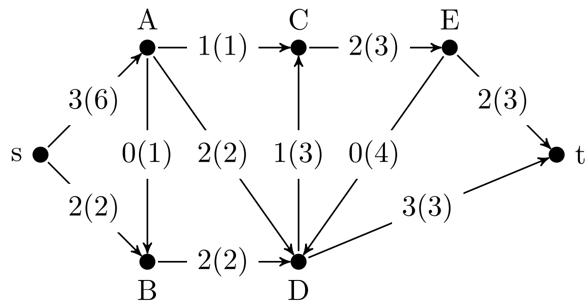
Definition 4. The quantity $\sum_{v \leftarrow s} f(s,v) - \sum_{u \to s} f(u,s) = \sum_{u \to t} f(u,t) - \sum_{v \leftarrow t} f(t,v)$ is known as the *value* $|f|$ of a flow.

Our question all along, of determining how much material we can route through a particular system, is thus no more and no less than the question of what the maximum value of a flow of a particular weighted graph can be.

At this point, an illustration perhaps is in order. Let us consider the following weighted graph with marked source and sink vertices, and capacities on the edges.



With some puzzling, we might start by routing 1 unit of flow along the $s \to A \to C \to E \to t$ route, 1 unit on the $s \to A \to D \to C \to E \to t$ route, 1 unit on the $s \to A \to D \to t$ route, and 2 units on the $s \to B \to D \to t$ route. Our total flow would be as such (with capacities in parentheses):



There are several observations to be made about this sample flow. First: relationships between inflows and outflows can be complicated, as at vertex D, where inflows of 2 and 2 correspond with outflows of 1 and 3. Another is that capacities are frequently underutilized, sometimes in nonintuitive ways: the route $s \to A \to B \to D \to E$ is a long path with a great deal of unused capacity, but the final leg of such a flow is not available. Lastly, there is the knowledge that this flow is in fact a flow of maximum value: no value greater than 5 can be possible. We may observe this final fact by dividing the graph into two "stages": consider s, A, and B to be vertices in the first stage, and C, D, E, and t in the second. Every route through the graph traverses between the stages, so no flow's value can exceed the total capacity of the edges from the first stage to the second, which is 5 (notice that in this example those three edges are completely utilized, which suggests that they are the bottleneck).

1.1 Cuts and Flows

We can in fact formalize this last observation, which turns out to be critical.

Definition 5. A *cut* in a weighted digraph with a source s and sink t is a partition of the vertices of the digraph into two sets S and T such that $s \in S$ and $t \in T$. The *capacity* of such a cut is

$$c(S,T) = \sum_{\substack{u \in S \\ v \in T \\ u \to v}} c(u,v)$$

Proposition 2. *If f is a flow in a weighted digraph G with a source s and sink t, and (S,T) is a cut in the same graph, then $|f| \leq c(S,T)$.*

Proof. We know that the difference between outflow and inflow at every vertex except s and t is zero, and at s the difference is exactly $|f|$, so we may write:

$$|f| = \sum_{u \in V(G) \setminus \{t\}} \left(\sum_{v \leftarrow u} f(u,v) - \sum_{v \to u} f(v,u) \right)$$

Dividing this up into sections in S and in $T \setminus \{t\}$, we get

$$|f| = \sum_{u \in S} \left(\sum_{v \leftarrow u} f(u,v) - \sum_{v \to u} f(v,u) \right) + \sum_{u \in T \setminus \{t\}} \left(\sum_{v \leftarrow u} f(u,v) - \sum_{v \to u} f(v,u) \right)$$

But since inflow and outflow is the same in every single point of T except t, we know the second term above will be zero. Now, in the first term, we shall divide up the possible values of v based on whether they are in S or in T and distribute:

$$|f| = \sum_{u\in S}\sum_{\substack{v\in S\\ v\leftarrow u}} f(u,v) - \sum_{u\in S}\sum_{\substack{v\in S\\ v\rightarrow u}} f(v,u) + \sum_{u\in S}\sum_{\substack{v\in T\\ v\leftarrow u}} f(u,v) - \sum_{u\in S}\sum_{\substack{v\in T\\ v\rightarrow u}} f(v,u)$$

Now let us note that both the first and second terms of this summation total the flow along edges *within* S, so they cancel out. The fourth term looks slightly problematic, but since $f(u,v) \geq 0$, all we simply need to do is use this expression as a lower bound:

$$\begin{aligned} |f| &= \sum_{u\in S}\sum_{\substack{v\in T\\ v\leftarrow u}} f(u,v) - \sum_{u\in S}\sum_{\substack{v\in T\\ v\rightarrow u}} f(v,u) \\ &\leq \sum_{u\in S}\sum_{\substack{v\in T\\ v\leftarrow u}} f(u,v) \\ &\leq \sum_{u\in S}\sum_{\substack{v\in T\\ v\leftarrow u}} c(u,v) = c(S,T) \end{aligned}$$

□

So we have seen that every cut is at least as large as every flow. This is an awfully familiar setup, and suggests that this structure is philosophically akin to other min-max results we have seen, such as antichains and chain partitions as governed by Dilworth's Theorem, vertex covers and matchings as governed by the König-Egerváry Theorem, and cutsets and independent paths as governed by Menger's Theorem. Indeed, we have a result analogous to those eminent theorems:

Theorem 1 (Max-Flow Min-Cut, a.k.a. Ford-Fulkerson/Elias-Feinstein-Shannon '56)**.** *In a weighted digraph G with a source s and sink t, there is a flow f and a cut (S,T) such that $|f| = c(S,T)$.*

Proof. Let f be a cut of maximum value (note: since the value of flows is bounded above by the capacity of an arbitrary cut, and since there are a finite number of different edges with a finite number of different capacities, such a cut is guaranteed to exist; there are peculiar infinite-analytic cases where such an assumption cannot be blithely made, but in a finite graph it is supportable). We shall construct a cut (S,T) guaranteed to have capacity of $|f|$ via a recursive procedure.

Let $s \in S$. Then, for each $x \in S$ we probe its neighbors y, putting y in S if either $x \to y$ with $f(x,y) < c(x,y)$ or if $y \to x$ with $f(y,x) > 0$. After it becomes impossible to add any more elements to S via this proceduce, let $T = V(G) \setminus S$. It shall be seen that $c(S,T) = |f|$.

First, in order for $c(S,T)$ to be computable at all, we must demonstrate that $t \in T$. Suppose, contrariwise, that $t \in S$. Then there is a sequence of steps whereby t was reached in order to be added to S, so we have a sequence of vertices $s = x_0, x_1, x_2, x_3, \dots x_{k-1}, x_k = t$ such that for each i, either $x_i \to x_{i+1}$ with $f(x_i,x_{i+1}) < c(x_i,x_{i+1})$ or if $x_i \leftarrow x_{i+1}$ with $f(x_{i+1},x_i) > 0$. For each such pair denote by q_i the positive quantity $c(x_i,x_{i+1}) - f(x_i,x_{i+1})$ or $f(x_{i+1},x_i)$ as appropriate, and let $q = \min q_i$. We can produce a flow f' which is identical to f except on the edges noted above: where $x_i \to x_{i+1}$, we let $f'(x_i,x_{i+1}) = f(x_i,x_{i+1}) + q$, and where $x_i \leftarrow x_{i+1}$, we let $f'(x_{i+1},x_i) = f(x_{i+1},x_i) - q$. In effect, the result of this procedure is to increase the flow on the undirected path from s to t determined by the x_i, treating backwards-facing edges as if they constituted negative flow. This can be shown to continue to satisfy the flow conditions fairly easily, and it will become clear, due to the modification in flow on the edge (s, x_1) or (x_1, s), that $|f'| = |f| + q$, violating maximality of f.

We saw in the proof of Proposition 2 that, regardless of which cut (S, T) is chosen, it is true that

$$|f| = \sum_{u \in S} \sum_{\substack{v \in T \\ v \leftarrow u}} f(u, v) - \sum_{u \in S} \sum_{\substack{v \in T \\ v \to u}} f(v, u)$$

However, by the construction of our S above, it must be the case that for any $u \in S$ and $v \notin S$, if $u \to v$, then $f(u, v) = c(u, v)$, and that if $v \to u$, then $f(v, u) = 0$. Thus the above can be rephrased as:

$$|f| = \sum_{u \in S} \sum_{\substack{v \in T \\ v \leftarrow u}} c(u, v) - \sum_{u \in S} \sum_{\substack{v \in T \\ v \to u}} 0 = c(S, T)$$

□