1 Tournaments

Definition 1. An orientation of the complete undirected graph is called a *tournament*.

The definition above is of a natural sort of real-world structure: if there are n competitors in a tournament conducted via a round-robin (i.e. a structure such that each competitor plays each other competitor exactly once), then this structure is a handy combinatorial way to record the wins and losses, directing each edge from the winner towards the loser.

The rather fuzzily-described real-world problem which naturally follows from this is: is there a sensible mapping from a tournament on n vertices to an ordering on an n-element set? Ideally we would have a total ordering, but doing so would be in many circumstances unfeasible: there is no total ordering we could in good faith associate with the directed cycle C_3, or in a symmetric structure on any odd number of vertices, so it behooves us to relax the question to a partial ordering.

What we mean by "sensible" above is admittedly rather vague, which takes this problem outside the realm of formal combinatorics, although it's a question still of interest at mathematics's intersection with the social sciences: economics and voting theory.

However, we note that cycles are really the only problem with a sensible ordering, so we might pay specific attention to the question of what sort of tournament has no cycles and thus an unambiguous ordering of the participants:

Proposition 1. *The acyclic tournament on n vertices is uniquely defined up to permutation of vertices.*

Proof. We shall prove this by induction on n. The case $n = 1$ is trivial.

By a result seen yesterday, an acyclic digraph has a vertex of zero indegree. Thus an acyclic tournament T on n vertices has a vertex v such that $d^-(v) = 0$. Since the edges among $V(T) - v$ are orientations of a complete graph on $n-1$ vertices, $T - v$ is a tournament on $n-1$ vertices, and it is acyclic, since it is a subgraph of an acyclic graph.

Since T is an orientation of a complete graph, there are $n - 1$ edges incident on v, and since $d^-(v) = 0$, they are all outbound edges. Thus, T can be specifically characterized as an acyclic tournament on $n - 1$ vertices, together with a vertex v, and edges from v to every other vertex of the graph.

However, by the inductive hypothesis, the acyclic tournament on $n - 1$ vertices is uniquely determined up to permutation. The selection of v is arbitrary up to permutation, so T itself is constructed from a series of steps with no choices performed on a unique smaller graph, and is thus itself unique. $\square$

The acyclic tournament is the only one which can be associated with the naïve ordering that $u \preceq v$ iff $u = v$ or $v \to u$; such an ordering requires that the tournament be transitive, and, as seen previously, transitive graphs must be acyclic.

There are other partial, and even occasionally total, orderings possible for association with even nontransitive tournaments. Here is a small sampling of common tournament orderings:

Win-loss record $u \preceq v$ if either $u = v$ or $d^-(u) < d^+(v)$. Distinct vertices of equal outdegree are incomparable.

Cycle-ambiguity If u and v lie on a directed cycle, u and v are incomparable. Otherwise, $u \preceq v$ iff $v \to u$.

Proof that win-loss record forms a partial ordering is fairly easy; showing that cycle-ambiguous edge-directed ordering is a partial ordering is a bit trickier, but the avoidance of cycles allows the transitive property to be justified.

One curious way to identify "leader" vertices in a graph comes from an approach of Landau's:

Theorem 1 (Landau, '53)**.** *Every tournament T has at least one vertex u such that every vertex in T is at a distance of at most 2 from u.*

Proof. Let u be a vertex of T of maximum outdegree (which may not uniquely determine u). Suppose u does not satisfy the above property. Then there is a vertex v which is not at a distance of 2 or less from u; since its distance is not zero, $u \neq v$. Since its distance from u is specifically not 1, it follows that $(u, v) \notin E(T)$, and, since some orientation of the edge $\{u, v\}$ must be in T, it follows that $v \to u$.

Since the distance from u to v is not 2, it additionally follows that there is no $w \neq u, v$ such that $u \to w \to v$. Thus, if $(u, w) \in E(T)$, then $(w, v) \notin E(T)$, and since $\{w, v\}$ must have some orientation, $(v, w) \in E(T)$. Thus, every terminus of an outgoing edge from u is also a terminus of an outgoing edge from v. In addition, u is itself the tail of an outgoing edge from v, and thus v has greater outdegree than u, contradicting maximality. □

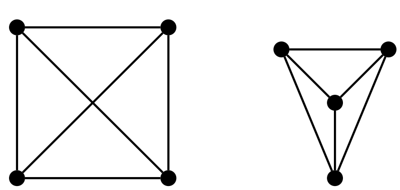
Note that while maximum outdegree guarantees the Landau property, the converse is not true, so this is a sense of leadership which is actually somewhat weaker than outdegree ordering.

2 Planarity

Up until now, we have mostly treated graphs as a combinatorial abstraction, and while we've represented them geometrically, we've thought of the geometry as a convenience to ourselves, rather than as a property of the graph in its own right. Now we shall consider the geometry of embedding a graph in the plane.

Definition 2. A *planar representation* of a graph G is a mapping of the vertices of G to distinct points in the plane, and of the edges to curves in the plane, such that the endpoints of curves correspond to the endpoints of the associated edges, and the edges do not intersect except at their endpoints. A graph is *planar* if it has a planar representation.

Note that many graphs have traditional representations that are nonplanar, but are, in fact, actually planar:



Above we see the "traditional" visual representation of K_4, which has intersecting edges and is thus nonplanar, but note that it may also be represented in the second way, which is planar, so K_4 is planar, even though some of its intuitive representations are not.

Proving things are planar can be notoriously geometrically fiddly. There are some easy examples though:

Proposition 2. *The cycle C_n is planar, for all n.*

Proof. Let the vertices and edges of C_n be associated with the vertices and edges of an n-gon. The edges are by definition nonintersecting, and terminate at the vertices as required. □

Proposition 3. *Every tree is planar.*

Proof. Denote $\Delta(T)$ by Δ for brevity. Choose a vertex v of a tree and position it at $(0,0)$. Let each of v's neighbors be at positions $(0,1)$, $(1,1)$, $(2,1)$, etc. For each u which is a neighbor of v with position $(k,1)$, let its neighbors except for its "parent" be at position $(k,2)$, $(k+\frac{1}{\Delta},2)$, $(k+\frac{2}{\Delta},2)$, $(k+\frac{3}{\Delta},2)$, and so forth. In general, let the unplaced neighbors of (k,r) be at positions $(k+\frac{i}{\Delta^r},r+1)$ for $0 \leq i < \Delta$. This is guaranteed to place vertices so that straight edges do not intersect (the full proof is tediously geometric). □

Proposition 4. *If G is a graph with planar finite components, G is planar.*

Proof. Since a planar projection can be arbitrarily assigned to a particular region of the plane via translations and dilations, let us consider for each component C_i a planar representation lying withing the square with corners $(0,0)$ and $(1,1)$. Now, let us apply to each C_i a horizontal translation of $2i$. Since the projections lie within nonoverlapping squares, none of their edges can overlap, and since each is a planar projection, no edges internal to a particular projection oveerlap. □

The most significant geometric result relating to planarity is actually most familiar as a fact from solid geometry:

Theorem 2 (Euler's Theorem)**.** *A convex poyhedron with v vertices, e edges, and f faces must satisfy $v+f-e=2$.*

A cursory examination shows that this is true for the tetrahedron ($v=4$, $e=6$, $f=4$), cube ($v=8$, $e=12$, $f=6$), octahedron ($v=6$, $e=12$, $f=8$), dodecahedron ($v=20$, $e=30$, $f=12$), and icosohedron ($v=12$, $e=30$, $f=20$). Somewhat less obvious is the fact that each of these polyhedra can be squashed into the plane, by removing one face, and stretching the others until they lie flat. So a polyhedron-related result can really be made explicit (and in some ways easier to prove) with plane figures.

Definition 3. A *face* of a planar representation of a graph is an open connected subset of the plane containing no edges, all of whose boundaries are edges.

Theorem 3 (Euler's Theorem Revisited)**.** *If G is a nonempty connected planar graph with a representation having $f(G)$ faces, then $|G|+f(G)-\|G\|=2$.*

Proof. We shall perform induction on $|G| = n$. The case $n = 1$ has 1 face and zero edges, so $|G| + f - \|G\| = 1 + 1 - 0 = 2$.

For our inductive step, by the structural theorem on connected graphs, there is a vertex v such that $G - v$ is connected. By the indictive hypothesis, $|G - v| + f(G - v) - \|G - v\| = 2$, so $(|G| - 1) + f(G - v) - (\|G\| - d_G(v)) = 2$. Thus, regardless of what projection is chosen for $G - v$, $f(G - v) = 3 + \|G\| - |G| - d_G(v)$.

Let us consider how many faces G has. v lies within some face of $G - v$, which cannot be a face of G. However, every face of $G - v$ except this one is guaranteed to be a face of G: if any face of $G - v$ intersected an edge incident on v, then by necessity it would intersect v as well.

We thus need only perform forensics on the effect introducing v has on the particular face containing v. We can order the edges coming out of v by sweeping counterclockwise from the positive horizontal and arbitrarily labeling v's incident edges as e_1, e_2, e_3, $\ldots$, e_k. We shall see that each of the k wedges so demarcated is a distinct face, so that $f(G) = f(G - v) + k - 1$, which would result in $f(G) = 2 + \|G\| - |G|$, proving Euler's Theorem.

Why would this be so? Consider, without lack of generality, the face delineated by the edges e_1 and e_2. Since $G - v$ is connected, there is a path from the endpoint of e_1 to the endpoint of e_2 not passing through v. Together with e_1 and e_2 themselves, this path constitutes a cycle incident on 3 points of the aforementioned face. The boundary of the face itself will also be such a cycle, and we have seen, since v appears only once in the cycle, between the edges e_1 and e_2, that no other e_i is on the boundary of the face, so the wedge e_i, e_{i+1} for $i \neq 1$ is a completely distinct face. $\square$